A Hirzebruch proportionality principle in Arakelov geometry

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Summary. We describe a tautological subring in the arithmetic Chow ring of bases of abelian schemes. Among the results are an Arakelov version of the Hirzebruch proportionality principle and a formula for a critical power of $\widehat{c}_1$ of the Hodge bundle.
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1 Introduction

The purpose of this note is to exploit some implications of a fixed point formula in Arakelov geometry when applied to the action of the (-1) involution on abelian schemes of relative dimension d. It is shown that the fixed point formula's statement in this case is equivalent to giving the values of arithmetic Pontrjagin classes of the Hodge bundle $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$, where these Pontrjagin classes are defined as polynomials in the arithmetic Chern classes defined by Gillet and Soulé. The resulting formula (see Theorem 3.4) is

$$\widehat{p}_k(\overline{E}) = (-1)^k \left(\frac{2\zeta'(1-2k)}{\zeta(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{j} - \frac{2\log 2}{1-4^{-k}} \right) (2k-1)!\, a(\mathrm{ch}(E)^{[2k-1]}) \tag{1}$$

with the canonical map a defined on classes of differential forms. When combined with the statement of the Gillet-Soulé's non-equivariant arithmetic Grothendieck-Riemann-Roch formula ([GS8],[Fal]), one obtains a formula for the class $\hat{c}_1^{1+d(d-1)/2}$ of the d-dimensional Hodge bundle in terms of topological classes and a certain special differential form γ (Theorem 5.1), which represents an Arakelov Euler class. Morally, this should be regarded as a formula for the height of complete cycles of codimension d in the moduli space (but the non-existence of such cycles for $d \geq 3$ has been shown by Keel and Sadun [KS]). Still it might serve as a model for the non-complete case. Finally we derive an Arakelov version of the Hirzebruch proportionality principle (not to be confused with its extension by Mumford [M]), namely a ring homomorphism from the Arakelov Chow ring $\mathrm{CH}^*(\overline{L}_{d-1})$ of Lagrangian Grassmannians to the arithmetic Chow ring of bases of abelian schemes $\widehat{\mathrm{CH}}^*(B)$ (Theorem 5.5):

Theorem 1.1. *Let S denote the tautological bundle on L_{d-1}. There is a ring homomorphism*

$$h\colon \mathrm{CH}^*(\overline{L}_{d-1})_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^*(B)_{\mathbb{Q}}/(a(\gamma))$$

with

$$h(\widehat{c}(\overline{S})) = \widehat{c}(\overline{E}) \left(1 + a\left(\sum_{k=1}^{d-1}(\frac{\zeta'(1-2k)}{\zeta(1-2k)} - \frac{\log 2}{1-4^{-k}})(2k-1)!\mathrm{ch}^{[2k-1]}(E)\right)\right)$$

and

$$h(a(c(\overline{S}))) = a(c(\overline{E}))\,.$$

In the last section we investigate the Fourier expansion of the Arakelov Euler class γ of the Hodge bundle on the moduli space of principally polarized abelian varieties.

A fixed point formula for maps from arithmetic varieties to Spec D has been proven by Roessler and the author in [KR1], where D is a regular arithmetic ring. In [KR2, Appendix] we described a conjectural generalization to flat equivariantly projective maps between arithmetic varieties over D. The missing ingredient to the proof of this conjecture was the equivariant version of Bismut's formula for the behavior of analytic torsion forms under the composition of immersions and fibrations [B4], i.e., a merge of [B3] and [B4]. This formula has meanwhile been shown by Bismut and Ma [BM].

There is insofar a gap in our proof of this result (Conjecture 3.2), as we give only a sketch. While our sketch is quite exhaustive and provides a rather complete guideline to an extension of a previous proof in [KR1] to the one required here, a fully written up version of the proof would still be basically a copy of [KR1] and thus be quite lengthly. This is not the subject of this article.

We work only with regular schemes as bases; extending these results to moduli stacks and their compactifications remains an open problem, as Arakelov geometry for such situations has not yet fully been developed. A corresponding Arakelov intersection ring has been established in [BKK] by Burgos, Kramer and Kühn, but the associated K-theory of vector bundles does not exist yet; see [MR] for associated conjectures. In particular one could search an analogue of the Hirzebruch-Mumford proportionality principle in Arakelov geometry. Van der Geer investigated the classical Chow ring of the moduli stack of abelian varieties and its compactifications [G] with a different method. The approach there to determine the tautological subring uses the non-equivariant Grothendieck-Riemann-Roch theorem applied to line bundles associated to theta divisors. Thus it might be possible to avoid the use of the fixed point formula in our situation by mimicking this method, possibly by extending the methods of Yoshikawa [Y]; but computing the occurring objects related to the theta divisor is presumably not easy.

Results extending some parts of an early preprint form of this article ([K2]) in the spirit of Mumford's extension of the proportionality principle have been conjectured in [MR]. That article also exploits the case in which more special automorphisms exist than the (-1) automorphism. Their conjectures and results are mainly generalizing Corollary 4.1.

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2 Torsion forms

Let $\pi\colon E^{1,0} \to B$ denote a d-dimensional holomorphic vector bundle over a complex manifold. Let Λ be a lattice subbundle of the underlying real vector bundle $E^{1,0}_{\mathbb{R}}$ of rank $2d$. Thus the quotient bundle $M := E^{1,0}/\Lambda \to B$ is a holomorphic fibration by tori Z. Let

$$\Lambda^* := \left\{\mu \in (E^{1,0}_{\mathbb{R}})^* \;\middle|\; \mu(\lambda) \in 2\pi\mathbb{Z} \quad \text{for all } \lambda \in \Lambda\right\}$$

denote the dual lattice bundle. Assume that $E^{1,0}$ is equipped with an Hermitian metric such that the volume of the fibers is constant. Any polarization induces such a metric.

Let N_V be the number operator acting on $\Gamma(Z, \Lambda^q T^{*0,1} Z)$ by multiplication with q. Let Tr_s denote the supertrace with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading on $\Lambda T^*B \otimes \mathrm{End}(\Lambda T^{*0,1}Z)$. Let ϕ denote the map acting on $\Lambda^{2p}T^*B$ as multiplication by $(2\pi i)^{-p}$. We write $\widetilde{\mathfrak{A}}(B)$ for $\widetilde{\mathfrak{A}}(B) := \bigoplus_{p\geq 0}(\mathfrak{A}^{p,p}(B)/(\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}))$, where $\mathfrak{A}^{p,p}(B)$ denotes the C^∞ differential forms of type (p,p) on B. We shall denote a vector bundle F together with an Hermitian metric h by $\overline{F}$. Then $\mathrm{ch}_g(\overline{F})$ shall denote the Chern-Weil representative of the equivariant Chern character associated to the restriction of (F, h) to the fixed point subvariety. Recall (see e.g. [B3]) also that $\mathrm{Td}_g(\overline{F})$ is the differential form

$$\mathrm{Td}_g(\overline{F}) := \frac{c_{\mathrm{top}}(\overline{F}^g)}{\sum_{k\geq 0}(-1)^k \mathrm{ch}_g(\Lambda^k \overline{F})} .$$

In [K1, Section 3], a superconnection A_t acting on the infinite-dimensional vector bundle $\Gamma(Z, \Lambda T^{*0,1}Z)$ over B has been introduced, depending on $t \in \mathbb{R}^+$. For a fibrewise acting holomorphic isometry g the limit

$$\lim_{t\to\infty} \phi \mathrm{Tr}_s g^* N_H e^{-A_t^2} =: \omega_\infty$$

exists and is given by the respective trace restricted to the cohomology of the fibers. The equivariant analytic torsion form $T_g(\pi, \overline{\mathcal{O}_M}) \in \widetilde{\mathfrak{A}}(B)$ was defined there as the derivative at zero of the zeta function with values in differential forms on B given by

$$-\frac{1}{\Gamma(s)}\int_0^\infty (\phi \mathrm{Tr}_s g^* N_H e^{-A_t^2} - \omega_\infty) t^{s-1}\, dt$$

for $\operatorname{Re} s > d$.

Theorem 2.1. *Let an isometry g act fibrewise with isolated fixed points on the fibration by tori $\pi: M \to B$. Then the equivariant torsion form $T_g(\pi, \mathcal{O}_M)$ vanishes.*

Proof. Let $f_\mu: M \to \mathbb{C}$ denote the function $e^{i\mu}$ for $\mu \in \Lambda^*$. As is shown in [K1, §5] the operator A_t^2 acts diagonally with respect to the Hilbert space decomposition

$$\Gamma(Z, \Lambda T^{*0,1} Z) = \bigoplus_{\mu \in \Lambda^*} \Lambda E^{*0,1} \otimes \{f_\mu\}\,.$$

As in [KR4, Lemma 4.1] the induced action by g maps a function f_μ to a multiple of itself if and only if $\mu = 0$ because g acts fixed point free on $E^{1,0}$ outside the zero section. In that case, f_μ represent an element in the cohomology. Thus the zeta function defining the torsion vanishes. □

Remark. As in [KR4, Lemma 4.1], the same proof shows the vanishing of the equivariant torsion form $T_g(\pi, \overline{\mathcal{L}})$ for coefficients in a g-equivariant line bundle $\overline{\mathcal{L}}$ with vanishing first Chern class.

We shall also need the following result of [K1] for the non-equivariant torsion form $T(\pi, \mathcal{O}_M) := T_{\mathrm{id}}(\pi, \mathcal{O}_M)$: Assume for simplicity that π is Kähler. Consider for $\operatorname{Re} s < 0$ the zeta function with values in $(d-1, d-1)$-forms on B

$$Z(s) := \frac{\Gamma(2d-s-1)\mathrm{vol}(M)}{\Gamma(s)(d-1)!} \sum_{\lambda \in \Lambda \setminus \{0\}} \Big(\frac{\overline{\partial}\partial}{4\pi i} \|\lambda^{1,0}\|^2\Big)^{\wedge(d-1)} \big(\|\lambda^{1,0}\|^2\big)^{s+1-2d}$$

where $\lambda^{1,0}$ denotes a lattice section in $E^{1,0}$. (In [K1] the volume is equal to 1.) Then the limit $\gamma := \lim_{s \to 0^-} Z'(0)$ exists and it transgresses the Chern-Weil form $c_d(\overline{E^{0,1}})$ representing the Euler class $c_d(E^{0,1})$

$$\frac{\overline{\partial}\partial}{2\pi i}\gamma = c_d(\overline{E^{0,1}})\,.$$

In [K1, Th. 4.1] the torsion form is shown to equal

$$T(\pi, \overline{\mathcal{O}_M}) = \frac{\gamma}{\mathrm{Td}(\overline{E^{0,1}})}$$

in $\widetilde{\mathfrak{A}}(B)$. The differential form γ was intensively studied in [K1].

3 Abelian schemes and the fixed point formula

We shall use the Arakelov geometric concepts and notation of [SABK] and [KR1]. In this article we shall only give a brief introduction to Arakelov geometry, and we refer to [SABK] for details. Let D be a regular arithmetic ring, i.e., a regular, excellent, Noetherian integral ring, together with a finite set $\mathcal{S}$ of ring monomorphisms of $D \to \mathbb{C}$, invariant under complex conjugation. We shall denote by $G := \mu_n$ the diagonalizable group scheme over D associated to $\mathbb{Z}/n\mathbb{Z}$. We choose once and for all a primitive n-th root of unity $\zeta_n \in \mathbb{C}$. Let $f\colon Y \to \mathrm{Spec}\ D$ be an equivariant arithmetic variety, i.e., a regular integral scheme, endowed with a μ_n-projective action over Spec D. The groups of n-th roots of unity acts on the d-dimensional manifold $Y(\mathbb{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to ζ_n.

We write f^{μ_n} for the map $Y_{\mu_n} \to \mathrm{Spec}\ D$ induced by f on the fixed point subvariety. Complex conjugation induces an antiholomorphic automorphism of $Y(\mathbb{C})$ and $Y_{\mu_n}(\mathbb{C})$, both of which we denote by F_∞. The space $\widetilde{\mathfrak{A}}(Y)$ is the sum over p of the subspaces of $\widetilde{\mathfrak{A}}^{p,p}(Y(\mathbb{C}))$ of classes of differential $(p,p)-$forms ω such that $F_\infty^*\omega = (-1)^p\omega$. Let $D^{p,p}(Y(\mathbb{C}))$ denote similarly the F_∞-equivariant currents as duals of differential forms of type $(d-p, d-p)$. It contains in particular the Dirac currents $\delta_{Z(\mathbb{C})}$ of p-codimensional subvarieties Z of Y.

Gillet-Soulé's arithmetic Chow ring $\widehat{\mathrm{CH}}^*(Y)$ is the quotient of the $\mathbb{Z}$-module generated by pairs (Z, g_Z) with Z an arithmetic subvariety of codimension p, $g_Z \in D^{p-1,p-1}(Y(\mathbb{C}))$ with $\frac{\partial\overline{\partial}}{2\pi i} g_Z + \delta_{Z(\mathbb{C})}$ being a smooth differential form by the submodule generated by the pairs $(\mathrm{div}\ f, -\log\|f\|^2)$ for rational functions f on Y. Let $\mathrm{CH}^*(Y)$ denote the classical Chow ring. Then there is an exact sequence in any degree p

$$\mathrm{CH}^{p,p-1}(Y) \xrightarrow{\rho} \widetilde{\mathfrak{A}}^{p-1,p-1}(Y) \xrightarrow{a} \widehat{\mathrm{CH}}^p(Y) \xrightarrow{\zeta} \mathrm{CH}^p(Y) \longrightarrow 0\,. \qquad (2)$$

For Hermitian vector bundles $\overline{E}$ on Y Gillet and Soulé defined arithmetic Chern classes $\widehat{c}_p(\overline{E}) \in \widehat{\mathrm{CH}}^p(Y)_{\mathbb{Q}}$.

By "product of Chern classes", we shall understand in this article any product of at least two equal or non-equal Chern classes of degree greater than 0 of a given vector bundle.

Lemma 3.1. *Let*

$$\widehat{\phi} = \sum_{j=0}^{\infty} a_j \widehat{c}_j + \text{products of Chern classes}$$

denote an arithmetic characteristic class with $a_j \in \mathbb{Q}$ *and* $a_j \neq 0$ *for* $j > 0$*. Assume that for a vector bundle* $\overline{F}$ *on an arithmetic variety* Y *we have* $\widehat{\phi}(\overline{F}) = m + a(\beta)$ *where* β *is a differential form on* $Y(\mathbb{C})$ *with* $\partial\overline{\partial}\beta = 0$ *and* $m \in \widehat{\mathrm{CH}}^0(Y)_{\mathbb{Q}}$*. Then*

$$\sum_{j=0}^{\infty} a_j \widehat{c}_j(\overline{F}) = m + a(\beta)\,.$$

Proof. We use induction. For the term in $\widehat{\mathrm{CH}}^0(Y)_{\mathbb{Q}}$, the formula is clear. Assume now for $k \in \mathbb{N}_0$ that

$$\sum_{j=0}^{k} a_j \widehat{c}_j(\overline{F}) = m + \sum_{j=0}^{k} a(\beta)^{[j]}\,.$$

Then $\widehat{c}_j(\overline{F}) \in a(\ker \partial\overline{\partial})$ for $1 \le j \le k$, thus products of these $\widehat{c}_j$'s vanish by [SABK, Remark III.2.3.1]. Thus the term of degree $k+1$ of $\widehat{\phi}(\overline{F})$ equals $a_{k+1}\widehat{c}_{k+1}(\overline{F})$. □

We define *arithmetic Pontrjagin classes* $\widehat{p}_j \in \widehat{\mathrm{CH}}^{2j}$ of arithmetic vector bundles by the relation

$$\sum_{j=0}^{\infty} (-z^2)^j \widehat{p}_j := \Big(\sum_{j=0}^{\infty} z^j \widehat{c}_j\Big)\Big(\sum_{j=0}^{\infty} (-z)^j \widehat{c}_j\Big)\,.$$

Thus,

$$\widehat{p}_j(\overline{F}) = (-1)^j \widehat{c}_{2j}(\overline{F} \oplus \overline{F}^*) = \widehat{c}_j^2(\overline{F}) + 2\sum_{l=1}^{j} (-1)^l \widehat{c}_{j+l}(\overline{F})\widehat{c}_{j-l}(\overline{F})$$

for an arithmetic vector bundle $\overline{F}$ (compare [MiS, §15]). Similarly to the construction of Chern classes via the elementary symmetric polynomials, the Pontrjagin classes can be constructed using the elementary symmetric polynomials in the squares of the variables. Thus many formulae for Chern classes have an easily deduced analogue for Pontrjagin classes. In particular, Lemma 3.1 holds with Chern classes replaced by Pontrjagin classes.

Now let Y, B be μ_n-equivariant arithmetic varieties over some fixed arithmetic ring D and let $\pi: Y \to B$ be a map over D, which is flat, μ_n-projective and smooth over the complex numbers. Fix a $\mu_n(\mathbb{C})$-invariant Kähler metric on $Y(\mathbb{C})$. We recall [KR1, Definition 4.1] extending the definition of Gillet-Soulé's arithmetic K_0-theory to the equivariant setting: Let $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}})$ be an equivariant Bott-Chern secondary class as introduced in [KR1, Th. 3.4]. The arithmetic equivariant Grothendieck group $\widehat{K}^{\mu_n}(Y)$ of Y is the sum of the abelian group $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ and the free abelian group generated by the equivariant isometry classes of Hermitian vector bundles, together with the following relations: For every short exact sequence $\overline{\mathcal{E}}: 0 \to E' \to E \to E'' \to 0$ and any equivariant metrics on E, E', and E'', we have the relation $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$ in $\widehat{K}^{\mu_n}(Y)$. We remark that $\widehat{K}^{\mu_n}(Y)$ has a natural ring structure. We denote the canonical map $\widetilde{\mathfrak{A}}(Y_{\mu_n}) \to \widehat{K}^{\mu_n}(Y)$ by a; the canonical trivial Hermitian line bundle $\overline{\mathcal{O}}$ shall often be denoted by 1.

If $\overline{E}$ is a π-acyclic (meaning that $R^k\pi_*E = 0$ if $k > 0$) μ_n-equivariant Hermitian bundle on Y, let $\pi_*\overline{E}$ be the direct image sheaf (which is locally free), endowed with its natural equivariant structure and L_2-metric. Consider the rule which associates the element $\pi_*\overline{E} - T_g(\pi, \overline{E})$ of $\widehat{K}_0^{\mu_n}(B)$ to every π-acyclic equivariant Hermitian bundle $\overline{E}$ and the element

$$\int_{Y(\mathbb{C})_g/B(\mathbb{C})_g} \mathrm{Td}_g(\overline{T\pi})\eta \quad \in \widetilde{\mathfrak{A}}(B_{\mu_n})$$

to every $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$. This rule induces a group homomorphism $\pi_!\colon \widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(B)$ ([KR2, Prop. 3.1]).

Let $\mathcal{R}$ be a ring as appearing in the statement of [KR1, Th. 4.4] (in the cases considered in this paper, we can choose $\mathcal{R} = D[1/2]$) and let $R(\mu_n)$ be the Grothendieck group of finitely generated projective μ_n-comodules. Let $\lambda_{-1}(E)$ denote the alternating sum $\sum_k (-1)^k \Lambda^k E$ of a vector bundle E. Consider the zeta function $L(\alpha, s) = \sum_{k=1}^{\infty} k^{-s}\alpha^k$ for Re $s > 1$, $|\alpha| = 1$. It has a meromorphic continuation to $s \in \mathbb{C}$ which shall be denoted by L, too. Then $L(-1, s) = (2^{1-s} - 1)\zeta(s)$ and the function

$$\widetilde{R}(\alpha, x) := \sum_{k=0}^{\infty} \left(\frac{\partial L}{\partial s}(\alpha, -k) + L(\alpha, -k) \sum_{j=1}^{k} \frac{1}{2j} \right) \frac{x^k}{k!}$$

defines the Bismut equivariant R-class of an equivariant holomorphic hermitian vector bundle $\overline{E}$ with $\overline{E}_{|X_g} = \sum_\zeta \overline{E}_\zeta$ as

$$R_g(\overline{E}) := \sum_{\zeta \in S^1} \left(\mathrm{Tr}\, \widetilde{R}\big(\zeta, -\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}\big) - \mathrm{Tr}\, \widetilde{R}\big(1/\zeta, \frac{\Omega^{\overline{E}_\zeta}}{2\pi i}\big) \right) .$$

The following result was stated as a conjecture in [KR2, Conj. 3.2].

Conjecture 3.2. Set

$$\mathrm{td}(\pi) := \frac{\lambda_{-1}(\pi^* \overline{N}^*_{B/B_{\mu_n}})}{\lambda_{-1}(\overline{N}^*_{Y/Y_{\mu_n}})} \Big(1 - a(R_g(N_{Y/Y_{\mu_n}})) + a(R_g(\pi^* N_{B/B_{\mu_n}}))\Big) .$$

Then the following diagram commutes

$$\begin{array}{ccc} \widehat{K}_0^{\mu_n}(Y) & \xrightarrow{\mathrm{td}(\pi)\rho'} & \widehat{K}_0^{\mu_n}(Y_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R} \\ \big\downarrow \pi_! & & \big\downarrow \pi_!^{\mu_n} \\ \widehat{K}_0^{\mu_n}(B) & \xrightarrow{\rho'} & \widehat{K}_0^{\mu_n}(B_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R} \end{array}$$

where ρ' denotes the restriction to the fixed point subscheme.

As this result is not the main aim of this paper, we only outline the proof; details shall appear elsewhere.

Sketch of the proof. As explained in [KR2, conjecture 3.2] the proof of the main statement of [KR1] was already written with this general result in mind and it holds without any major change for this situation, when using the generalization of Bismut's equivariant immersion formula for the holomorphic torsion ([KR1, Th. 3.11]) to torsion forms. The latter has now been established by Bismut and Ma [BM]. The proof in [KR1] holds when using [BM] instead of [KR1, Th. 3.11] and [KR2, Prop. 3.1] instead of [KR1, Prop. 4.3].

Also one has to replace in sections 5, 6.1 and 6.3 the integrals over Y_g, X_g etc. by integrals over Y_g/B_g, X_g/B_g, while replacing the maps occurring there by corresponding relative versions. As direct images can occur as non-locally-free coherent sheaves, one has to consider at some steps suitable resolutions of vector bundles such that the higher direct images of the vector bundles in this resolution are locally free as e.g. on [Fal, p. 74]. □

Let $f\colon B \to \operatorname{Spec} D$ denote a quasi-projective arithmetic variety and let $\pi\colon Y \to B$ denote a principally polarized abelian scheme of relative dimension d. For simplicity, we assume that the volume of the fibers over $\mathbb{C}$ is scaled to equal 1; it would be 2^d for the metric induced from the polarization. We shall explain the effect of rescaling the metric later (after Theorem 5.1). Set $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$. This bundle $E = \mathbf{Lie}(Y/B)^*$ is the Hodge bundle. Then by [BBM, Prop. 2.5.2], the full direct image of $\mathcal{O}$ under π is given by $R^\bullet\pi_*\mathcal{O} = \Lambda^\bullet E^*$ and the relative tangent bundle is given by $T\pi = \pi^*E^*$. By similarly representing the cohomology of the fibres Y/B by translation-invariant differential forms, one shows that these isomorphisms induce isometries if and only if the volume of the fibres equals 1 (e.g. as in [K1, Lemma 3.0]), thus

$$\overline{R^\bullet\pi_*\mathcal{O}} = \Lambda^\bullet\overline{E}^* \tag{3}$$

and

$$\overline{T\pi} = \pi^*\overline{E}^* \,. \tag{4}$$

See also [FC, Th. VI.1.1], where these properties are extended to toroidal compactifications. For an action of $G = \mu_n$ on Y Conjecture 3.2 combined with the arithmetic Grothendieck-Riemann-Roch theorem in all degrees for π^G states (analogous to [KR1, section 7.4]):

Theorem 3.3.

$$\widehat{\mathrm{ch}}_G(\overline{R^\bullet\pi_*\mathcal{O}}) - a(T_g(\pi_{\mathbb{C}}, \overline{\mathcal{O}})) = \pi_*^G\big(\widehat{\mathrm{Td}}_G(\overline{T\pi})(1 - a(R_g(T\pi_{\mathbb{C}})))\big) \,.$$

As in [KR1], $G = \mu_n$ is used as the index for equivariant arithmetic classes, while the chosen associated automorphism g over the points at infinity is used for objects defined there. We shall mainly consider the case where π^G is actually a smooth covering, Riemannian over $\mathbb{C}$; thus the statement of the arithmetic Grothendieck-Riemann-Roch is in fact very simple in this case. We obtain the equation

$$\widehat{\mathrm{ch}}_G(\Lambda^\bullet\overline{E}^*) - a(T_g(\pi_{\mathbb{C}}, \overline{\mathcal{O}})) = \pi_*^G\big(\widehat{\mathrm{Td}}_G(\pi^*\overline{E}^*)(1 - a(R_g(\pi^* E_{\mathbb{C}}^*)))\big) \,.$$

Using the equation

$$\widehat{\mathrm{ch}}_G(\Lambda^\bullet \overline{E}^*) = \frac{\widehat{c}_{\mathrm{top}}(\overline{E}^G)}{\widehat{\mathrm{Td}}_G(\overline{E})}$$

this simplifies to

$$\frac{\widehat{c}_{\mathrm{top}}(\overline{E}^G)}{\widehat{\mathrm{Td}}_G(\overline{E})} - a(T_g(\pi_{\mathbb{C}}, \overline{\mathcal{O}})) = \widehat{\mathrm{Td}}_G(\overline{E}^*)(1 - a(R_g(E^*_{\mathbb{C}})))\pi^G_* \pi^* 1\,,$$

or, using that $a(\ker \bar\partial\partial)$ is an ideal of square zero,

$$\widehat{c}_{\mathrm{top}}(\overline{E}^G)(1 + a(R_g(E^*_{\mathbb{C}}))) - a(T_g(\pi_{\mathbb{C}}, \overline{\mathcal{O}})\mathrm{Td}_g(\overline{E}_{\mathbb{C}})) = \widehat{\mathrm{Td}}_G(\overline{E})\widehat{\mathrm{Td}}_G(\overline{E}^*)\pi^G_* \pi^* 1\,. \tag{5}$$

Remarks. 1) If G acts fibrewise with isolated fixed points (over $\mathbb{C}$), by Theorem 2.1 the left hand side of equation (5) is an element of $\widehat{\mathrm{CH}}^0(B)_{\mathbb{Q}(\zeta_n)} + a(\ker \partial\bar\partial)$. Set for an equivariant bundle F in analogy to the classical $\hat{A}$-class

$$\widehat{A}_g(F) := \mathrm{Td}_g(F) \exp(-\frac{c_1(F) + \mathrm{ch}_g(F)^{[0]}}{2}) \tag{6}$$

and let $\widehat{\widehat{A}_G}$ denote the corresponding arithmetic class (an unfortunate clash of notations); in particular $\widehat{A}_g(F^*) = (-1)^{\mathrm{rk}(F/F^G)}\widehat{A}_g(F)$. For isolated fixed points, by comparing the components in degree 0 in equation (5) one obtains

$$\pi^G_* \pi^* 1 = (-1)^d (\widehat{A}_g(E)^{[0]})^{-2}$$

and thus by Theorem 2.1

$$1 + a(R_g(E^*_{\mathbb{C}})) = \left(\frac{\widehat{\widehat{A}_G}(\overline{E})}{\widehat{A}_g(E)^{[0]}} \right)^2 . \tag{7}$$

(compare [KR4, Prop. 5.1]). Both sides can be regarded as products over the occurring eigenvalues of g of characteristic classes of the corresponding bundles E_ζ. One can wonder whether the equality holds for the single factors, similar to [KR4]. Related work is announced by Maillot and Roessler in [MR].

2) If $G(\mathbb{C})$ does not act with isolated fixed points, then the right hand side vanishes, $c_{\mathrm{top}}(E^G)$ vanishes and we find

$$\widehat{c}_{\mathrm{top}}(\overline{E}^G) = a\big(T_g(\pi_{\mathbb{C}}, \overline{\mathcal{O}})\mathrm{Td}_g(\overline{E}_{\mathbb{C}})\big)\,. \tag{8}$$

As was mentioned in [K1, eq. (7.8)], one finds in particular

$$\widehat{c}_d(\overline{E}) = a(\gamma)\,. \tag{9}$$

For this statement we need Gillet-Soulé's arithmetic Grothendieck-Riemann-Roch [GS8] in all degrees, while the above statements use this theorem only in degree 0. The full result was stated in [S, section 4]; a proof of an analogue statement is given in [R2, section 8]. Another proof was sketched in [Fal] using a possibly different direct image. If one wants to avoid the use of this strong result, one can at least show the existence of some $(d-1, d-1)$ differential form γ' with $\widehat{c}_d(\overline{E}) = a(\gamma')$ the following way: The analogue proof of equation (9) in the classical algebraic Chow ring $\mathrm{CH}^*(B)$ using the classical Riemann-Roch-Grothendieck Theorem shows the vanishing of $c_d(E)$. Thus by the exact sequence

$$\widetilde{\mathfrak{A}}^{d-1,d-1}(B) \xrightarrow{a} \widehat{\mathrm{CH}}^d(B) \xrightarrow{\zeta} \mathrm{CH}^d(B) \longrightarrow 0$$

we see that (9) holds with some form γ'.

Now we restrict ourself to the action of the automorphism (-1). We need to assume that this automorphism corresponds to a μ_2-action. This condition can always be satisfied by changing the base Spec D to Spec $D[\frac{1}{2}]$ ([KR1, Introduction] or [KR4, section 2]).

Theorem 3.4. *Let $\pi: Y \to B$ denote a principally polarized abelian scheme of relative dimension d over an arithmetic variety B. Set $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$. Then the Pontrjagin classes of $\overline{E}$ are given by*

$$\widehat{p}_k(\overline{E}) = (-1)^k \left(\frac{2\zeta'(1-2k)}{\zeta(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{j} - \frac{2\log 2}{1-4^{-k}} \right) (2k-1)!\, a(\mathrm{ch}(E)^{[2k-1]}) \ . \tag{10}$$

The log 2-term actually vanishes in the arithmetic Chow ring over Spec $D[1/2]$.

Remark. The occurrence of R-class-like terms in Theorem 3.4 makes it very unlikely that there is an easy proof of this result which does not use arithmetic Riemann-Roch-Theorems. This is in sharp contrast to the classical case over $\mathbb{C}$, where the analogues formulae are a trivial topological result: The underlying real vector bundle of $E_{\mathbb{C}}$ is flat, as the period lattice determines a flat structure. Thus the topological Pontrjagin classes $p_j(E_{\mathbb{C}})$ vanish.

Proof. Let $Q(z)$ denote the power series in z given by the Taylor expansion of

$$4(1+e^{-z})^{-1}(1+e^z)^{-1} = \frac{1}{\cosh^2 \frac{z}{2}}$$

at $z = 0$. Let $\widehat{Q}$ denote the associated multiplicative arithmetic characteristic class. Thus by definition for $G = \mu_2$

$$4^d \widehat{\mathrm{Td}}_G(\overline{E}) \widehat{\mathrm{Td}}_G(\overline{E}^*) = \widehat{Q}(\overline{E})$$

and $\widehat{Q}$ can be represented by Pontrjagin classes, as the power series Q is even. Now we can apply Lemma 3.1 for Pontrjagin classes to equation (5) of equation (7). By a formula by Cauchy [Hi3, §1, eq. (10)], the summand of $\widehat{Q}$

consisting only of single Pontrjagin classes is given by taking the Taylor series in z at $z=0$ of

$$Q(\sqrt{-z})\frac{d}{dz}\frac{z}{Q(\sqrt{-z})} = \frac{\frac{d}{dz}(z\cosh^2\frac{\sqrt{-z}}{2})}{\cosh^2\frac{\sqrt{-z}}{2}} = 1 + \frac{\sqrt{-z}}{2}\tanh\frac{\sqrt{-z}}{2} \tag{11}$$

and replacing every power z^j by $\widehat{p}_j$. The bundle $\overline{E}^G$ is trivial, hence $\widehat{c}_{\mathrm{top}}(\overline{E}^G) = 1$. Thus by equation (5) with $\pi_*^G\pi^*1 = 4^d$ we obtain

$$\sum_{k=1}^{\infty}\frac{(4^k-1)(-1)^{k+1}}{(2k-1)!}\zeta(1-2k)\widehat{p}_k(\overline{E}) = -a(R_g(E_{\mathbb{C}}))\,.$$

The function $\widetilde{R}(\alpha,x)$ by which the Bismut equivariant R-class is constructed satisfies for $\alpha=-1$ the relation

$$\widetilde{R}(-1,x)-\widetilde{R}(-1,-x) = \sum_{k=1}^{\infty}\Big[\,(4^k-1)\Big(2\zeta'(1-2k)+\zeta(1-2k)\sum_{j=1}^{2k-1}\frac{1}{j}\Big) \\ -2\log 2\cdot 4^k\zeta(1-2k)\Big]\cdot\frac{x^{2k-1}}{(2k-1)!}\,. \tag{12}$$

Thus we finally obtain the desired result. □

The first Pontrjagin classes are given by

$$\widehat{p}_1 = -2\widehat{c}_2+\widehat{c}_1^2\,,\quad \widehat{p}_2 = 2\widehat{c}_4-2\widehat{c}_3\widehat{c}_1+\widehat{c}_2^2\,,\quad \widehat{p}_3 = -2\widehat{c}_6+2\widehat{c}_5\widehat{c}_1-2\widehat{c}_4\widehat{c}_2+\widehat{c}_3^2\,.$$

In general, $\widehat{p}_k = (-1)^k 2\widehat{c}_{2k}$+products of Chern classes. Thus knowing the Pontrjagin classes allows us to express the Chern classes of even degree by the Chern classes of odd degree.

Corollary 3.5. *The Chern-Weil form representing the total Pontrjagin class vanishes (except in degree 0):*

$$c(\overline{E}\oplus\overline{E}^*) = 1\,,\quad \textit{i.e.,}\quad \det(1+(\Omega^E)^{\wedge 2}) = 1$$

for the curvature Ω^E of the Hodge bundle. The Pontrjagin classes in the algebraic Chow ring $\mathrm{CH}(B)$ vanish:

$$c(E\oplus E^*) = 1\,.$$

Proof. These facts follow from applying the forget-functors $\omega\colon\widehat{\mathrm{CH}}(B)\to\mathfrak{A}(B(\mathbb{C}))$ and $\zeta\colon\widehat{\mathrm{CH}}(B)\to\mathrm{CH}(B)$. □

The first fact can also be deduced by "linear algebra", e.g. using the Mathai-Quillen calculus, but it is not that easy. The second statement was obtained in [G, Th. 2.5] using the non-equivariant Grothendieck-Riemann-Roch theorem and the geometry of theta divisors.

4 A K-theoretical proof

The Pontrjagin classes form one set of generators of the algebra of even classes; another important set of generators is given by $(2k)!$ times the Chern character in even degrees $2k$. We give the value of these classes below. Let U denote the additive characteristic class associated to the power series

$$U(x) := \sum_{k=1}^{\infty} \left(\frac{\zeta'(1-2k)}{\zeta(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{2j} - \frac{\log 2}{1-4^{-k}} \right) \frac{x^{2k-1}}{(2k-1)!}$$

and let d again denote the relative dimension of the abelian scheme.

Corollary 4.1. *The part of* $\widehat{\mathrm{ch}}(\overline{E})$ *in* $\widehat{\mathrm{CH}}^{\mathrm{even}}(B)_{\mathbb{Q}}$ *is given by the formula*

$$\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{even}]} = d - a(U(E))\,.$$

Proof. The part of $\widehat{\mathrm{ch}}(\overline{E})$ of even degree equals

$$\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{even}]} = \frac{1}{2}\widehat{\mathrm{ch}}(\overline{E} \oplus \overline{E}^*)\,,$$

thus it can be expressed by Pontrjagin classes. More precisely by Newton's formulae ([Hi3, §10.1]),

$$(2k)!\widehat{\mathrm{ch}}^{[2k]} - \widehat{p}_1 \cdot (2k-2)!\widehat{\mathrm{ch}}^{[2k-2]} + \cdots + (-1)^{k-1}\widehat{p}_{k-1}2!\widehat{\mathrm{ch}}^{[2]} = (-1)^{k+1}k\widehat{p}_k$$

for $k \in \mathbb{N}$. As products of the arithmetic Pontrjagin classes vanish in $\widehat{\mathrm{CH}}(Y)_{\mathbb{Q}}$ by Lemma 3.4, we thus observe that the part of $\widehat{\mathrm{ch}}(\overline{E})$ in $\widehat{\mathrm{CH}}^{\mathrm{even}}(Y)_{\mathbb{Q}}$ is given by

$$\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{even}]} = d + \sum_{k>0} \frac{(-1)^{k+1}\widehat{p}_k(\overline{E})}{2(2k-1)!}\,.$$

Thus the result follows from Lemma 3.4. □

As Harry Tamvakis pointed out to the author, a similar argument is used in [T, section 2] and its predecessors.

Now we show how to deduce Corollary 4.1 (and thus the equivalent Theorem 3.4) using only Conjecture 3.2 without combining it with the arithmetic Grothendieck-Riemann-Roch Theorem as in Theorem 3.3. Of course the structure of the proof shall not be too different as the Grothendieck-Riemann-Roch Theorem was very simple in this case; but the following proof is quite instructive as it provides a different point of view on the resulting characteristic classes. We shall use the λ-ring structure on $\widehat{K}$ constructed in [R1].

Conjecture 3.2 applied to the abelian scheme $\pi\colon Y \to B$ provides the formula

$$\pi_!\overline{\mathcal{O}} = \pi_!^{\mu_2} \frac{1 - a(R_g(N_{Y/Y_{\mu_n}}))}{\lambda_{-1}(\overline{N}^*_{Y/Y_{\mu_n}})}\,.$$

In our situation, $\overline{N_{Y/Y_{\mu_n}}} = \overline{T\pi}$. Combining this with the fundamental equations (3), (4) and Theorem 2.1 yields

$$\lambda_{-1}\overline{E}^* = \pi_!^{\mu_2}\pi^*\frac{1-a(R_g(E^*))}{\lambda_{-1}\overline{E}}$$

and using the projection formula we find

$$\lambda_{-1}\overline{E\oplus E^*} = 4^d\big(1-a(R_g(E^*))\big)\,.$$

Let $\overline{E}'$ denote the vector bundle E equipped with the trivial μ_2-action. Now one can deduce from this that $\overline{E'\oplus E'^*}$ itself has the form $2d+a(\eta)$ with a $\bar\partial\partial$-closed form η: Apply the Chern character to both sides. Then use equation (11) and Lemma 3.1 to deduce by induction that all Chern classes of $\overline{E'\oplus E'^*}$ are in $a(\ker\bar\partial\partial)$. Thus using the fact that the arithmetic Chern character is an isomorphism up to torsion ([GS3, Th. 7.3.4]) $\overline{E'\oplus E'^*} = 2d+a(\eta)$ with $a(\eta)$ having even degrees, and $\overline{E\oplus E^*} = (2d+a(\eta))\otimes(-1)$ in $\widehat{K}^{\mu_2}(B)_{\mathbb{Q}}$. One could use the γ-filtration instead to deduce this result; it would be interesting to find a proof which does not use any filtration.

For a $\beta\in\widetilde{\mathfrak{A}}^{p,p}(B)$, the action of the λ-operators can be determined as follows: The action of the k-th Adams operator is given by $\psi^k a(\beta) = k^{p+1}a(\beta)$ ([GS3, p. 235]). Then with $\psi_t := \sum_{k>0} t^k\psi^k$, $\lambda_t := \sum_{k\geq 0} t^k\lambda^k$ the Adams operators are related to the λ-operators via

$$\psi_t(x) = -t\frac{d}{dt}\log\lambda_{-t}(x)$$

for $x\in\widehat{K}^{\mu_n}(B)$. As $\psi_t(a(\beta)) = \mathrm{Li}_{-1-p}(t)a(\beta)$ with the polylogarithm Li, we find for $\beta\in\ker\bar\partial\partial$

$$\lambda_t(a(\beta)) = 1-\mathrm{Li}_{-p}(-t)a(\beta)$$

or $\lambda^k a(\beta) = -(-1)^k k^p a(\beta)$ ($\mathrm{Li}_{-p}(\frac{t}{t-1})$ is actually a polynomial in t; in this context this can be regarded as a relation coming from the γ-filtration). In particular $\lambda_{-1}a(\beta) = 1-\zeta(-p)a(\beta)$, and $\lambda_{-1}(a(\beta)\otimes(-1)) = \lambda_1 a(\beta)\otimes 1 = (1+(1-2^{p+1})\zeta(-p)a(\beta))\otimes 1$ in $\widehat{K}^{\mu_2}\otimes_{R_{\mu_2}}\mathbb{C}$.

By comparing

$$\lambda_{-1}\big(a(\eta)\otimes(-1)\big) = a\Big(\sum_{k>0}\zeta(1-2k)(1-4^k)\eta^{[2k-1]}\Big)\otimes 1 = a\big(R_{-1}(E^*)\big)\otimes 1$$

we finally derive $a(\eta) = a\big(-2U(E)\big)$ and thus

$$\overline{E'\oplus E'^*} = 2d-2a(U(E))\,.$$

In other words the Hermitian vector bundle $\overline{E'\oplus E'^*}$ equals the $2d$-dimensional trivial bundle plus the class of differential forms given by $U(E)$ in $\widehat{K}^{\mu_2}\otimes_{R_{\mu_2}}\mathbb{C}$. From this Corollary 4.1 follows.

5 A Hirzebruch proportionality principle and other applications

The following formula can be used to express the height of complete subvarieties of codimension d of the moduli space of abelian varieties as an integral over differential forms.

Theorem 5.1. *There is a real number $r_d \in \mathbb{R}$ and a Chern-Weil form $\phi(\overline{E})$ on $B_{\mathbb{C}}$ of degree $(d-1)(d-2)/2$ such that*

$$\widehat{c}_1^{1+d(d-1)/2}(\overline{E}) = a\big(r_d \cdot c_1^{d(d-1)/2}(E) + \phi(\overline{E})\gamma\big) .$$

The form $\phi(\overline{E})$ is actually a polynomial with integral coefficients in the Chern forms of $\overline{E}$. See Corollary 5.6 for a formula for r_d.

Proof. Consider the graded ring R_d given by $\mathbb{Q}[u_1, \dots, u_d]$ divided by the relations

$$\big(1 + \sum_{j=1}^{d-1} u_j\big)\big(1 + \sum_{j=1}^{d-1} (-1)^j u_j\big) = 1 , \quad \text{and} \quad u_d = 0 \tag{13}$$

where u_j shall have degree j ($1 \leq j \leq d$). This ring is finite dimensional as a vector space over $\mathbb{Q}$ with basis

$$u_{j_1} \cdots u_{j_m} , \qquad 1 \leq j_1 < \cdots < j_m < d \quad , 1 \leq m < d .$$

In particular, any element of R_d has degree $\leq \frac{d(d-1)}{2}$. As the relation (13) is verified for $u_j = \widehat{c}_j(\overline{E})$ up to multiples of the Pontrjagin classes and $\widehat{c}_d(\overline{E})$, any polynomial in the $\widehat{c}_j(\overline{E})$'s can be expressed in terms of the $\widehat{p}_j(\overline{E})$'s and $\widehat{c}_d(\overline{E})$ if the corresponding polynomial in the u_j's vanishes in R_d.

Thus we can express $\widehat{c}_1^{1+d(d-1)/2}(\overline{E})$ as the image under a of a topological characteristic class of degree $d(d-1)/2$ plus γ times a Chern-Weil form of degree $(d-1)(d-2)/2$. As any element of degree $d(d-1)/2$ in R_d is proportional to $u_1^{d(d-1)/2}$, the Theorem follows. □

Any other arithmetic characteristic class of $\overline{E}$ vanishing in R_d can be expressed in a similar way.

Example 5.2. We shall compute $\widehat{c}_1^{1+d(d-1)/2}(\overline{E})$ explicitly for small d. Define topological cohomology classes r_j by $\widehat{p}_j(\overline{E}) = a(r_j)$ via Theorem 3.4. For $d = 1$, clearly

$$\widehat{c}_1(\overline{E}) = a(\gamma) .$$

In the case $d = 2$ we find by the formula for $\widehat{p}_1$

$$\widehat{c}_1^2(\overline{E}) = a(r_1 + 2\gamma) = a\Big[(-1 + \frac{8}{3}\log 2 + 24\zeta'(-1))c_1(E) + 2\gamma\Big] .$$

Combining the formulae for the first two Pontrjagin classes we get

$$\widehat{p}_2 = 2\widehat{c}_4 - 2\widehat{c}_3\widehat{c}_1 + \frac{1}{4}\widehat{c}_1^4 - \frac{1}{2}\widehat{c}_1^2\widehat{p}_1 + \frac{1}{4}\widehat{p}_1^2\,.$$

Thus for $d=3$ we find, using $c_3(E)=0$ and $c_1^2(E)=2c_2(E)$,

$$\begin{aligned}\widehat{c}_1^4(\overline{E}) &= a(2c_1^2(E)r_1 + 4r_2 + 8c_1(E)\gamma)\\ &= a\Big[(-\frac{17}{3} + \frac{48}{5}\log 2 + 48\zeta'(-1) - 480\zeta'(-3))c_1^3(E) + 8c_1(\overline{E})\gamma\Big]\,.\end{aligned}$$

For $d=4$ one obtains

$$\begin{aligned}\widehat{c}_1^7(\overline{E}) = a\Big[\ & 64c_2(E)c_3(E)r_1 - (8c_1(E)c_2(E) + 32c_3(E))r_2 + 64c_1(E)r_3\\ & +16\big(7c_1(\overline{E})c_2(\overline{E}) - 4c_3(\overline{E})\big)\gamma\Big]\,.\end{aligned}$$

As in this case $\mathrm{ch}(E)^{[1]} = c_1(E)$, $3!\mathrm{ch}(E)^{[3]} = -c_1^3(E)/2 + 3c_3(E)$ and $5!\mathrm{ch}(E)^{[5]} = c_1^5(E)/16$, we find

$$\begin{aligned}\widehat{c}_1^7(\overline{E}) = a\Big[&\Big(-\frac{1063}{60} + \frac{1520}{63}\log 2 + 96\zeta'(-1) - 600\zeta'(-3) + 2016\zeta'(-5)\Big)c_1^5(E)\\ &+16(7c_1(\overline{E})c_2(\overline{E}) - 4c_3(\overline{E}))\gamma\Big]\,.\end{aligned}$$

For $d=5$ on gets

$$\begin{aligned}\widehat{c}_1^{11}(\overline{E}) = a\Big[\ & 2816\,\gamma\,c_2\,(3\,c_1\,c_3 - 8\,c_4) + c_1^{10}\,\Big(-\frac{104611}{2520} + \frac{113632}{2295}\log(2)\\ & -3280\zeta'(-7) + 2352\zeta'(-5) - 760\zeta'(-3) + 176\zeta'(-1)\Big)\Big]\,,\end{aligned}$$

and for $d=6$

$$\begin{aligned}\widehat{c}_1^{16}(\overline{E}) = a\Big[&425984\,\gamma\,(11\,c_1\,c_2\,c_3\,c_4 - 91\,c_2\,c_3\,c_5) + 40\,c_1\,c_4\,c_5)\\ &+c_1^{15}\,\Big(-\frac{3684242}{45045} + \frac{3321026752}{37303695}\log(2) + \frac{36096}{13}\zeta'(-9)\\ &-\frac{526080}{143}\zeta'(-7) + \frac{395136}{143}\zeta'(-5) - \frac{136320}{143}\zeta'(-3) + \frac{3264}{11}\zeta'(-1)\Big)\Big].\end{aligned}$$

Remark. We shall shortly describe the effect of rescaling the metric for the characteristic classes described above. By the multiplicativity of the Chern character and using $\widehat{\mathrm{ch}}(\mathcal{O}, \alpha|\cdot|^2) = 1 - a(\log\alpha)$, $\widehat{\mathrm{ch}}(\overline{E})$ changes by

$$\log\alpha \cdot a(\mathrm{ch}(E))$$

when multiplying the metric on E^* by a constant $\alpha \in \mathbb{R}^+$ (or with a function $\alpha \in C^\infty(B(\mathbb{C}), \mathbb{R}^+)$). Thus, we observe that in our case $\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{odd}]}$ is invariant under rescaling on E^* and we get an additional term

$$\log\alpha \cdot a\big(\mathrm{ch}(E)^{[\mathrm{odd}]}\big)$$

on the right hand side in Corollary 4.1, when the volume of the fibers equals α^d instead of 1. Thus the right hand side of Theorem 3.4 gets an additional term

$$\frac{(-1)^{k+1}\log\alpha}{2(2k-1)!}\,a\big(\mathrm{ch}(E)^{[2k-1]}\big)\,.$$

Similarly,

$$\hat{c}_d(\overline{E}) = a(\gamma) + \log\alpha\cdot a(c_{d-1}(E))$$

for the rescaled metric. In Theorem 5.1, we obtain an additional

$$\log\alpha\cdot a\Big(\frac{d(d-1)+2}{2}\cdot c_1^{d(d-1)/2}(E)\Big)$$

on the right hand side and this shows

$$\phi(E)c_{d-1}(E) = \frac{d(d-1)+2}{2}\,c_1^{d(d-1)/2}(E)\,. \tag{14}$$

Alternatively, one can show the same formulae by investigating directly the Bott-Chern secondary class of $R\pi_*\mathcal{O}$ for the metric change.

Assume that the base space Spec D equals Spec $\mathcal{O}_K[\frac{1}{2}]$ for a number field K. We consider the push forward map

$$\widehat{\deg}\colon \widehat{\mathrm{CH}}(B) \longrightarrow \widehat{\mathrm{CH}}^1\big(\mathrm{Spec}(\mathcal{O}_K[\frac{1}{2}])\big) \longrightarrow \widehat{\mathrm{CH}}^1\big(\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])\big) \cong \mathbb{R}/(\mathbb{Q}\log 2)\,,$$

where the last identification contains the traditional factor $\frac{1}{2}$.

As Keel and Sadun [KS] have shown by proving a conjecture by Oort, the moduli space of principally polarized complex abelian varieties does not have any projective subvarieties of codimension d, if $d \geq 3$. Thus the following two corollaries have a non-empty content only for $d = 2$. Still it is likely that they serve as models for similar results for non-projective subvarieties in an extended Arakelov geometry in the spirit of [BKK]. For that reason we state them together with the short proof.

Using the definition

$$h(B) := \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\,\hat{c}_1^{1+\dim B_{\mathbb{C}}}(\overline{E}_{|B})$$

of the *global height* (thus defined modulo rational multiples of $\log 2$ in this case) of a projective arithmetic variety we find:

Corollary 5.3. *If* $\dim B_{\mathbb{C}} = \frac{d(d-1)}{2}$ *and* B *is projective, then the (global) height of* B *with respect to* $\det\overline{E}$ *is given by*

$$h(B) = \frac{r_d}{2}\cdot\deg B + \frac{1}{2}\int_{B_{\mathbb{C}}}\phi(\overline{E})\gamma\,.$$

with deg *denoting the algebraic degree.*

Let $\alpha(E,\Lambda,\omega^E)\in\bigwedge^* T^*B$ be a differential form associated to bundles of principally polarized abelian varieties (E,Λ,ω^E) (with Hodge bundle E, lattice Λ and polarization form ω^E) in a functorial way: If $f\colon B''\to B$ is a holomorphic map and $(f^*E,f^*\Lambda,f^*\omega^E)$ the induced bundle over B'', then $\alpha(f^*E,f^*\Lambda,f^*\omega^E)=f^*\alpha(E,\Lambda,\omega^E)$; in other words, α shall be a modular form. Choose an open cover (U_i) of B such that the bundle trivializes over U_i. To define the Hecke operator $T(p)$ for p prime, associated to the group $\mathrm{Sp}(n,\mathbb{Z})$, consider on U_i the set $\mathcal{L}(p)$ of all maximal sublattices $\Lambda'\subset\Lambda_{|U_i}$ such that ω^E takes values in $p\mathbb{Z}$ on Λ'. The sums

$$T(p)\alpha(E,\Lambda,\omega^E)_{|U_i}:=\sum_{\Lambda'\in\mathcal{L}(p)}\alpha(E,\Lambda',\frac{\omega^E}{p})$$

patch together to a globally defined differential form on B. Note that the set $\mathcal{L}(p)$ may be identified with the set of all maximal isotropic subspaces (Lagrangians) $\Lambda'/p\Lambda$ of the symplectic vector space $(\Lambda/p\Lambda,\omega^E)$ over $\mathbb{F}_p$.

Let B' be a disjoint union of abelian schemes with one connected component for each $\Lambda'\in\mathcal{L}(p)$ such that the Hodge bundle over each connected component over Spec $\mathbb{C}$ is isomorphic to the hodge bundle $E(\mathbb{C})$ over $B(\mathbb{C})$, but the period lattice and polarization form are given by Λ' and ω^E/p.

Corollary 5.4. *For B as in Corollary 5.3 set $h'(B):=\frac{h(B)}{(\dim B_{\mathbb{C}}+1)\deg B}$. The height of B and B' are related by*

$$h'(B')=h'(B)+\frac{p^d-1}{p^d+1}\cdot\frac{\log p}{2}\ .$$

Proof. For this proof we need that γ is indeed the form determined by the arithmetic Riemann-Roch Theorem in all degrees (compare equation (9)). The action of Hecke operators on γ was investigated in [K1, Section 7]. In particular it was shown that

$$T(p)\gamma=\prod_{j=1}^{d}(p^j+1)\left(\gamma+\frac{p^d-1}{p^d+1}\log p\cdot c_{d-1}(\overline{E})\right)\ .$$

The action of Hecke operators commutes with multiplication by a characteristic class, as the latter are independent of the period lattice in E. Thus by Corollary 5.3 the height of B' is given by

$$h(B')=$$
$$\prod_{j=1}^{d}(p^j+1)\left(\frac{r_d}{2}\cdot\deg B_{\mathbb{C}}+\frac{1}{2}\int_{B_{\mathbb{C}}}\phi(\overline{E})\gamma+\frac{p^d-1}{p^d+1}\frac{\log p}{2}\int_{B_{\mathbb{C}}}\phi(\overline{E})c_{d-1}(E)\right).$$

Combining this with equation (14) gives the result. □

Similarly one obtains a formula for the action of any other Hecke operator using the explicit description of its action on γ in [K1, equation (7.4)].

The choice of B' is modeled after the action of the Hecke operator $T(p)$ on the intersection cohomology on moduli of abelian varieties, as described in [FC, chapter VII.3], where B should be regarded as a subvariety of the moduli space and B' as representing its image under $T(p)$ in the intersection cohomology. This action is only defined over Spec $\mathbb{Z}[1/p]$ though. As $\widehat{\mathrm{CH}}^1(\mathrm{Spec}\ \mathbb{Z}[1/p]) = \mathbb{R}/(\mathbb{Q}\cdot\log p)$, the additional term in the above formula would disappear for this base.

Now we are going to formulate an Arakelov version of Hirzebruch's proportionality principle. In [Hi2, p. 773] it is stated as follows: Let G/K be a non-compact irreducible Hermitian symmetric space with compact dual G'/K and let $\Gamma \subset G$ be a cocompact subgroup such that $\Gamma\backslash G/K$ is a smooth manifold. Then there is an ring monomorphism

$$h: H^*(G'/K,\mathbb{Q}) \to H^*(\Gamma\backslash G/K,\mathbb{Q})$$

such that $h(c(TG'/K)) = c(TG/K)$ (and similar for other bundles F', F corresponding to K-representation V', V dual to each other). This implies in particular that Chern numbers on G'/K and $\Gamma\backslash G/K$ are proportional [Hi1, p. 345]. Now in our case think for the moment about B as the moduli space of principally polarized abelian varieties of dimension d. Its projective dual is the Lagrangian Grassmannian L_d over Spec $\mathbb{Z}$ parametrizing maximal isotropic subspaces in symplectic vector spaces of dimension $2d$ over any field, $L_d(\mathbb{C}) = \mathrm{Sp}(d)/\mathrm{U}(d)$. But as the moduli space is a non-compact quotient, the proportionality principle must be altered slightly by considering Chow rings modulo certain ideals corresponding to boundary components in a suitable compactification. For that reason we consider the Arakelov Chow group $\mathrm{CH}^*(\overline{L}_{d-1})$ with respect to the canonical Kähler metric on L_{d-1}, which is the quotient of $\mathrm{CH}^*(\overline{L}_d)$ modulo the ideal $(\widehat{c}_d(\overline{S}), a(c_d(\overline{S})))$ with $\overline{S}$ being the tautological bundle on L_d, and we map it to $\widehat{\mathrm{CH}}^*(B)/(a(\gamma))$. Here L_{d-1} shall be equipped with the canonical symmetric metric. For the Hermitian symmetric space L_{d-1}, the Arakelov Chow ring is a subring of the arithmetic Chow ring $\widehat{\mathrm{CH}}(L_{d-1})$ ([GS2, 5.1.5]) such that the quotient abelian group depends only on $L_{d-1}(\mathbb{C})$. Instead of dealing with the moduli space, we continue to work with a general regular base B.

The Arakelov Chow ring $\mathrm{CH}^*(\overline{L}_{d-1})$ has been investigated by Tamvakis in [T]. Consider the graded commutative ring

$$\mathbb{Z}[\widehat{u}_1,\ldots,\widehat{u}_{d-1}] \oplus \mathbb{R}[u_1,\ldots,u_{d-1}]$$

where the ring structure is such that $\mathbb{R}[u_1,\ldots,u_{d-1}]$ is an ideal of square zero. Let $\widehat{R}_d$ denote the quotient of this ring by the relations

$$\big(1+\sum_{j=1}^{d-1} u_j\big)\big(1+\sum_{j=1}^{d-1}(-1)^j u_j\big) = 1$$

and

$$\Big(1+\sum_{k=1}^{d-1}\widehat{u}_k\Big)\Big(1+\sum_{k=1}^{d-1}(-1)^k\widehat{u}_k\Big)$$
$$= 1-\sum_{k=1}^{d-1}\Big(\sum_{j=1}^{2k-1}\frac{1}{j}\Big)(2k-1)!\,\mathrm{ch}^{[2k-1]}(u_1,\ldots,u_{d-1}) \tag{15}$$

where $\mathrm{ch}(u_1,\ldots,u_{d-1})$ denotes the Chern character polynomial in the Chern classes, taken of $u_1,\ldots,u_{d-1}$. Then by [T, Th. 1], there is a ring isomorphism $\Phi\colon \widehat{R}_d \to \mathrm{CH}^*(\overline{L}_{d-1})$ with $\Phi(\widehat{u}_k)=\widehat{c}_k(\overline{S}^*)$ and $\Phi(u_k)=a(c_k(\overline{S}^*))$. The Chern character term in (15), which strictly speaking should be written as $(0,\mathrm{ch}^{[2k-1]}(u_1,\ldots,u_{d-1}))$, is thus mapped to

$$a\big(\mathrm{ch}^{[2k-1]}(c_1(\overline{S}^*),\ldots,c_{d-1}(\overline{S}^*))\big)\,.$$

Theorem 5.5. *There is a ring homomorphism*

$$h\colon \mathrm{CH}^*(\overline{L}_{d-1})_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{CH}}^*(B)_{\mathbb{Q}}/(a(\gamma))$$

with

$$h(\widehat{c}(\overline{S}))=\widehat{c}(\overline{E})\left(1+a\left(\sum_{k=1}^{d-1}(\frac{\zeta'(1-2k)}{\zeta(1-2k)}-\frac{\log 2}{1-4^{-k}})(2k-1)!\mathrm{ch}^{[2k-1]}(E)\right)\right)$$

and

$$h\big(a(c(\overline{S}))\big)=a(c(\overline{E}))\,.$$

Note that S^* and E are ample. One could as well map $a(c(\overline{S}^*))$ to $a(c(\overline{E}))$, but the correction factor for the arithmetic characteristic classes would have additional harmonic number terms.

Remark. For $d \leq 6$ one can in fact construct such a ring homomorphism which preserves degrees. Still this seems to be a very unnatural thing to do. This is thus in remarkable contrast to the classical Hirzebruch proportionality principle.

Proof. When writing the relation (15) as

$$\widehat{c}(\overline{S})\widehat{c}(\overline{S}^*)=1+a(\epsilon_1)$$

and the relation in Theorem 3.4 as

$$\widehat{c}(\overline{E})\widehat{c}(\overline{E}^*)=1+a(\epsilon_2)$$

we see that a ring homomorphism h is given by

$$h(\widehat{c}_k(\overline{S})) = \sqrt{\frac{1+h(a(\epsilon_1))}{1+a(\epsilon_2)}}\,\widehat{c}_k(\overline{E}) = \big(1+\frac{1}{2}h(a(\epsilon_1)) - \frac{1}{2}a(\epsilon_2)\big)\,\widehat{c}_k(\overline{E})$$

(where h on $\mathrm{im}(a)$ is defined as in the Theorem). Here the factor $1+\frac{1}{2}h(a(\epsilon_1))-\frac{1}{2}a(\epsilon_2)$ has even degree, and thus

$$h\big(\widehat{c}_k(\overline{S}^*)\big) = \sqrt{\frac{1+h(a(\epsilon_1))}{1+a(\epsilon_2)}}\,\widehat{c}_k(\overline{E}^*)$$

which provides the compatibility with the cited relations. □

Remarks. 1) Note that this proof does not make any use of the remarkable fact that $h(a(\epsilon_1^{[k]}))$ and $a(\epsilon_2^{[k]})$ are proportional forms for any degree k.

2) It would be favorable to have a more direct proof of Theorem 5.5, which does not use the description of the tautological subring. The R-class-like terms suggest that one has to use an arithmetic Riemann-Roch Theorem somewhere in the proof; one could wonder whether one could obtain the description of $\mathrm{CH}^*(\overline{L}_{d-1})$ by a method similar to section 3. Also, one might wonder whether the statement holds for other symmetric spaces. Our construction relies on the existence of a universal proper bundle with a fibrewise acting non-trivial automorphism; thus it shall not extend easily to other cases.

In particular Tamvakis' height formula [T, Th. 3] provides a combinatorial formula for the real number r_d occurring in Theorem 5.1. Replace each term $\mathcal{H}_{2k-1}$ occurring in [T, Th. 3] by

$$-\frac{2\zeta'(1-2k)}{\zeta(1-2k)} - \sum_{j=1}^{2k-1}\frac{1}{j} + \frac{2\log 2}{1-4^{-k}}$$

and divide the resulting value by half of the degree of L_{d-1}. Using Hirzebruch's formula

$$\deg L_{d-1} = \frac{(d(d-1)/2)!}{\prod_{k=1}^{d-1}(2k-1)!!}$$

for the degree of L_{d-1} (see [Hi1, p. 364]) and the $\mathbb{Z}_+$-valued function $g^{[a,b]_{d-1}}$ from [T] counting involved combinatorial diagrams, we obtain

Corollary 5.6. *The real number r_d occurring in Theorem 5.1 is given by*

$$r_d = \frac{2^{1+(d-1)(d-2)/2}\prod_{k=1}^{d-1}(2k-1)!!}{(d(d-1)/2)!} \cdot \sum_{k=0}^{d-2}\left(-\frac{2\zeta'(-2k-1)}{\zeta(-2k-1)} - \sum_{j=1}^{2k+1}\frac{1}{j} + \frac{2\log 2}{1-4^{-k-1}}\right) \cdot \sum_{b=0}^{\min\{k,d-2-k\}} (-1)^b 2^{-\delta_{b,k}} g^{[k-b,b]_{d-1}}\ ,$$

where $\delta_{b,k}$ is Kronecker's δ.

One might wonder whether there is a "topological" formula for the height of locally symmetric spaces similar to [KK, Theorem 8.1]. Comparing the fixed point height formula [KK, Lemma 8.3] with the Schubert calculus expression [T, Th. 3] for the height of Lagrangian Grassmannians, one finds

$$\sum_{\epsilon_1,\ldots,\epsilon_{d-1}\in\{\pm 1\}} \frac{1}{\prod_{i\leq j}(\epsilon_i i+\epsilon_j j)}$$
$$\sum_{\ell=1}^{\frac{d(d-1)}{2}} \sum_{i\leq j} \frac{\left(\sum \epsilon_\nu \nu\right)^{\frac{d(d-1)}{2}} - \left(\sum \epsilon_\nu \nu\right)^{\frac{d(d-1)}{2}-\ell+1}\left(\sum \epsilon_\nu \nu - (2-\delta_{ij}(\epsilon_i i+\epsilon_j j)\right)^\ell}{2\ell(\epsilon_i i+\epsilon_j j)}$$
$$= \sum_{k=0}^{d-2}\left(\sum_{j=1}^{2k-1}\frac{1}{j}\right)\sum_{b=0}^{\min\{k,d-2-k\}}(-1)^b 2^{-\delta_{b,k}} g^{[k-b,b]_{d-1}} .$$

In [G, Th. 2.5] van der Geer shows that R_d embeds into the (classical) Chow ring $\mathrm{CH}^*(\mathcal{M}_d)_{\mathbb{Q}}$ of the moduli stack $\mathcal{M}_d$ of principally polarized abelian varieties. Using this result one finds

Lemma 5.7. *Let B be a regular finite covering of the moduli space $\mathcal{M}_d$ of principally polarized abelian varieties of dimension d. Then for any non-vanishing polynomial expression $p(u_1,\ldots,u_{d-1})$ in R_d,*

$$h\big(p(\widehat{c}_1(\overline{S}),\ldots,\widehat{c}_{d-1}(\overline{S}))\big) \notin \operatorname{im} a\,.$$

In particular, h is non-trivial in all degrees. Furthermore, h is injective iff $a(c_1(E)^{d(d-1)/2}) \neq 0$ in $\widehat{\mathrm{CH}}^{d(d-1)/2+1}(B)_{\mathbb{Q}}/(a(\gamma))$.

The need for a regular covering in our context is an unfortunate consequence of the Arakelov geometry of stacks not yet being fully constructed. Eventually this problem might get remedied. Until then one can resort to base changes to ensure the existence of regular covers as e.g. the moduli space of p.p. abelian varieties with level-n structure for $n\geq 3$ over Spec $\mathbb{Z}[1/n, e^{2\pi i/n}]$ ([FC, chapter IV.6.2c]).

Proof. Consider the canonical map $\zeta : \widehat{\mathrm{CH}}^*(B)_{\mathbb{Q}}/(a(\gamma)) \to \mathrm{CH}^*(B)_{\mathbb{Q}}$. Then

$$\zeta\big(h(p(\widehat{c}_1(\overline{S}),\ldots,\widehat{c}_{d-1}(\overline{S})))\big) = p\big(c_1(E),\ldots,c_{d-1}(E)\big)\,,$$

and the latter is non-vanishing according to [G, Th. 1.5]. This proves the first assertion.

If $a(c_1(E)^{d(d-1)/2}) \neq 0$ in $\widehat{\mathrm{CH}}^{d(d-1)/2+1}(B)_{\mathbb{Q}}/(a(\gamma))$, then by the same induction argument as in the proof of [G, Th. 2.5] R_d embeds in $a(\ker\bar\partial\partial)$. Finally, by [T, Th. 2] any element z of $\widehat{R}_d$ can be written in a unique way as a linear combination of

$$\widehat{u}_{j_1} \cdots \widehat{u}_{j_m} \quad \text{and} \quad u_{j_1} \cdots u_{j_m}, \quad \text{with} \quad 1 \le j_1 < \cdots < j_m < d\,,\ 1 \le m < d\,.$$

Thus if $z \notin \operatorname{im} a$, then $h(z) \neq 0$ follows by van der Geer's result, and if $z \in \operatorname{im} a \setminus \{0\}$, then $h(z) \neq 0$ follows by embedding $R_d \otimes \mathbb{R}$. □

Using the exact sequence (2), the condition in the Lemma is that the cohomology class $c_1(E)^{d(d-1)/2}$ should not be in the image of the Beilinson regulator.

Finally by comparing Theorem 5.1 with Kühn's result [Kü, Theorem 6.1] (see also Bost [Bo]), we conjecture that the analogue of Theorem 5.5 holds in a yet to be developed Arakelov intersection theory with logarithmic singularities, extending the methods of [Kü], [BKK], as described in [MR]. I.e. there should be a ring homomorphism to the Chow ring of the moduli space of abelian varieties

$$h\colon \mathrm{CH}^*(\overline{L}_d)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^*(\mathcal{M}_d)_{\mathbb{Q}}$$

extending the one in Theorem 5.5, and γ should provide the Green current corresponding to $\hat{c}_d(\overline{E})$. This would imply

Conjecture 5.8. For an Arakelov intersection theory with logarithmic singularities, extending the methods of [Kü], the height of a moduli space $\mathcal{M}_d$ over $\operatorname{Spec} \mathbb{Z}$ of principally polarized abelian varieties of relative dimension d is given by

$$h(\mathcal{M}_d) = \frac{r_{d+1}}{2} \deg(\mathcal{M}_d)\,.$$

The factor $1/2$ is caused by the degree map in Arakelov geometry.

6 The Fourier expansion of the Arakelov Euler class of the Hodge bundle

In this section we shall further investigate the differential form γ which played a prominent role in the preceding results. We adapt most notations from [K1]. In particular we use as the base space the Siegel upper half space

$$\mathfrak{H}_n := \left\{ Z = X + iY \in \operatorname{End}(\mathbb{C}^d) \;\middle|\; {}^tZ = Z\,,\ Y > 0 \right\},$$

which is the universal covering of the moduli space of principally polarized abelian varieties. Due to an unavoidable clash of notations, we are forced here to use the letters Z and Y again. Choose the trivial $\mathbb{C}^d$-bundle over $\mathfrak{H}_n$ as the holomorphic vector bundle E and define the lattice Λ over a point $Z \in \mathfrak{H}_d$ as

$$\Lambda_{|Z} := (Z, \mathrm{id})\mathbb{Z}^{2n}$$

where (Z, id) denotes a $\mathbb{C}^{d \times 2d}$-matrix. The polarization defines a Kähler form on E; the associated metric is given by

$$\|Zr + s\|^2_{|Z} = {}^t(Zr + s)Y^{-1}\overline{(Zr + s)} \qquad \text{for } r, s \in \mathbb{Z}^n\,.$$

(one might scale the metric by a constant factor 1/2 to satisfy the condition $\mathrm{vol}(Z) = 1$. The torsion form is invariant under this scaling). The crucial ingredient in the construction of γ in [K1] was a series $\bar{\beta}_t$ depending on real parameters $t, b \in \mathbb{R}$, such that the Epstein zeta function $Z(s)$ with $\gamma = Z'(0)$ can be constructed as the Mellin transform of the b-linear term of $\bar{\beta}_t$. More precisely,

$$Z(s) := -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\frac{d}{db}_{|b=0} \bar{\beta}_t + c_{n-1}(\bar{E}) \right) dt \,,$$

which also leads to other expressions for γ in terms of $\bar{\beta}$. We derive the Fourier expansion for γ by applying the Poisson summation formula to a lattice of half the maximal rank in the Epstein zeta function defining the torsion form. This leaves us with two infinite series which converge at $s = 0$, and another Epstein zeta function for a lattice of half the previous rank. By iterating this procedure $\frac{\log 2d}{\log 2}$ times one can actually gain a convergent series expression for γ; compare [E, §8], where a similar procedure with $2d$ steps is described.

Set $C := \frac{1}{\pi} Y^{-1}(1 - \frac{1}{2\pi i b} \mathrm{Re}\, \Omega^E)$, and $D := \frac{1}{\pi} Y^{-1} \frac{-i}{2\pi i b} \mathrm{Im}\, \Omega^E$. Thus ${}^tC = C$, ${}^tD = -D$. Then by [K1, eq. (6.0)],

$$\begin{aligned} \bar{\beta}_t &= \left(\frac{-b}{\pi t} \right)^d \sum_{\lambda \in \Lambda} \exp\left(-\frac{1}{t} \langle \lambda^{1,0}, (1 + \frac{i}{2\pi b} \Omega^E) \lambda^{0,1} \rangle \right) \\ &= \left(\frac{-b}{\pi t} \right)^d \sum_{r,u \in \mathbb{Z}^d} \exp\left(-\frac{\pi}{t} {}^t(Zr + u)(C + D)(\bar{Z} r + u) \right) . \end{aligned}$$

Now let B be a symmetric integral $d \times d$-matrix. The space $\mathfrak{b}$ of such matrices embeds into $\mathrm{Sp}(d, \mathbb{Z})$ via

$$B \mapsto \begin{pmatrix} \mathrm{id} & B \\ 0 & \mathrm{id} \end{pmatrix} .$$

The induced action of $B \in \mathfrak{b}$ on $\mathfrak{H}$ is given by $Z \mapsto Z + B$. As $\bar{\beta}_t$ is $\mathrm{Sp}(d, \mathbb{Z})$-invariant, it thus has a Fourier decomposition on the torus $\mathfrak{H}/\mathfrak{b}$. Notice that the space $\mathfrak{c}$ of frequencies does not equal $\mathfrak{b}$ but is the space

$$\mathfrak{c} = \left\{ \frac{1}{2}(B + {}^t B) \;\middle|\; B \in \mathfrak{gl}(d, \mathbb{Z}) \right\}$$

of symmetric matrices integral along the diagonal and half-integral off the diagonal.

Using the Poisson summation formula applied to $u \in \mathbb{Z}^d$ we find for $\bar{\beta}_{t|Z}$ at $Z = X + iY$

$$\bar{\beta}_t = \left(\frac{-b}{\pi t} \right)^d \sum_{r,u \in \mathbb{Z}^d} \exp\left(-\frac{\pi}{t} {}^t(Zr + u)(C + D)(\bar{Z} r + u) \right)$$

$$= \left(\frac{-b}{\pi t}\right)^d \sum_{r,u\in\mathbb{Z}^d} \exp\Big(- \frac{\pi}{t} {}^t(Xr+u)C(Xr+u)$$
$$- \frac{\pi}{t} {}^t rYCYr - \frac{2\pi i}{t} {}^t rYD(Xr+u)\Big)$$
$$= \left(\frac{-b}{\pi\sqrt{t}}\right)^d \sum_{r,\hat{u}\in\mathbb{Z}^d} \frac{1}{\sqrt{\det C}} \exp\Big(- \pi t {}^t\hat{u}C^{-1}\hat{u} - 2\pi i {}^t\hat{u}Xr$$
$$- \frac{\pi}{t} {}^t rY(C - DC^{-1}D)Yr - 2\pi {}^t rYDC^{-1}\hat{u}\Big).$$

For any symmetric $A \in \mathbb{R}^{d\times d}$ and $M = \frac{1}{2}(r\cdot{}^t u + u\cdot{}^t r)$ we have $\langle M, A\rangle = \operatorname{Tr} M^t A = {}^t rAu$. Thus the Fourier coefficient of $e^{-2\pi i\langle M,X\rangle}$ for $M \in \mathfrak{c}$ equals

$$\sum \left(\frac{-b}{\pi\sqrt{t}}\right)^d \frac{1}{\sqrt{\det C}}$$
$$\cdot \exp\left(-\pi t {}^t uC^{-1}u - \frac{\pi}{t} {}^t rY(C - DC^{-1}D)Yr - 2\pi {}^t rYDC^{-1}u\right).$$

In particular the occurring frequency matrices M in the Fourier decomposition are among the matrices in $\mathfrak{c}$ which have at most two non-zero eigenvalue. Note that

$$C - DC^{-1}D = C(\mathrm{Id} - C^{-1}DC^{-1}D) = C(\mathrm{Id} - C^{-1}D)(\mathrm{Id} + C^{-1}D)$$
$$= (C - D)C^{-1}(C + D) = {}^t(C + D)C^{-1}(C + D) \qquad (16)$$

and in particular for $a \in \mathbb{R}^d$

$${}^t a(C - DC^{-1}D)^{-1}a = {}^t a(C + D)^{-1}(C \pm D)^t(C + D)^{-1}a = {}^t a(C \mp D)^{-1}a$$

(this value does not depend on the choice of $\pm$), or

$$2(C - DC^{-1}D)^{-1} = (C + D)^{-1} + (C - D)^{-1}.$$

6.1 The coefficients of the non-constant terms

Proposition 6.1. *Two vectors $r, u \in \mathbb{R}^d \setminus \{0\}$ are uniquely determined by the matrix*

$$M := \frac{1}{2}(r\cdot{}^t u + u\cdot{}^t r)$$

up to order and multiplication by a constant.

Proof. Assume first that u and r are not colinear. The two non-vanishing eigenvalues of M are given by

$$\lambda_{1,2} = \frac{1}{2}\big(\langle r, u\rangle \pm \|r\|\|u\|\big)$$

with corresponding eigenvectors $v_{1,2} = c_{1,2}(\|r\|u \pm \|u\|r)$ with $c_{1,2} \in \mathbb{R} \setminus \{0\}$ arbitrary. In fact,

$$Mv_{1,2} = \frac{c_{1,2}}{2}\big(\|r\|r\langle u,u\rangle \pm \|u\|r\langle r,u\rangle + \|r\|u\langle r,u\rangle \pm \|u\|u\langle r,r\rangle\big) = \lambda_{1,2}v_{1,2}\,.$$

Now $\|v_{1,2}\|^2 = \pm 4c_{1,2}^2\|r\|\|s\|\lambda_{1,2}$ and thus

$$\frac{v_{1,2}}{\|v_{1,2}\|}\sqrt{|\lambda_{1,2}|} = \pm\frac{1}{2}\left(\sqrt{\frac{\|r\|}{\|u\|}}u \pm \sqrt{\frac{\|u\|}{\|r\|}}r\right)$$

Without loss of generality we may assume the sign to be positive; we then have

$$\frac{v_1}{\|v_1\|}\sqrt{|\lambda_1|} + \frac{v_2}{\|v_2\|}\sqrt{|\lambda_2|} = \sqrt{\frac{\|r\|}{\|u\|}}u$$

and

$$\frac{v_1}{\|v_1\|}\sqrt{|\lambda_1|} - \frac{v_2}{\|v_2\|}\sqrt{|\lambda_2|} = \sqrt{\frac{\|u\|}{\|r\|}}r\,.$$

Thus all possible sets $\{u,r\}$ of solutions are given in terms of M by

$$\Big\{\{c(\frac{v_1}{\|v_1\|}\sqrt{|\lambda_1|} + \frac{v_2}{\|v_2\|}\sqrt{|\lambda_2|}), \frac{1}{c}(\frac{v_1}{\|v_1\|}\sqrt{|\lambda_1|} - \frac{v_2}{\|v_2\|}\sqrt{|\lambda_2|})\}\ \Big|\ c \in \mathbb{R}, c \neq 0\Big\}\,.$$

In the case r,u colinear the eigenvalue λ_2 vanishes and the proof remains the same with this simplification. □

Remarks. 1) Note that $\lambda_1 > 0$ and $\lambda_2 \leq 0$.

2) There is a simpler formula for r and u up to two possibilities in every coordinate: Necessarily one diagonal element of M is non-zero, say M_{11}. By solving the system of quadratic equation $2M_{1j} = r_1u_j + r_ju_1$, one finds up to the scaling constant

$$r_j = M_{1j} \pm \sqrt{M_{1j}^2 - M_{11}M_{jj}}\,.$$

Alas determining the $\pm$-choice in every coordinate is not easy.

3) In our case, the condition $r,s \in \mathbb{Z}^d$ implies that for every $M \in \mathfrak{c}$ there are primitive vectors $r_0, u_0 \in \mathbb{Z}^d$ and $c \in \mathbb{Z}^+$ such that all possible sets $\{r,u\}$ are given by $\{\{kr_0, c/k\cdot u_0\}|k \in \mathbb{Z}, k|c\}$.

Using the Taylor expansion of $(1-x)^{-1}$ at $x=0$, we find for the term in the exponential function in $\bar\beta_{t,M}$ with $r = kr_0$, $u = cu_0/k$

$$\begin{aligned} &-\pi t\,{}^t uC^{-1}u - \frac{\pi}{t}{}^t rY(C - DC^{-1}D)Yr - 2\pi\,{}^t rYDC^{-1}u \\ &= -\frac{\pi^2c^2}{k^2}t\,{}^t u_0Yu_0 - \frac{k^2}{t}{}^t r_0Yr_0 + \frac{t}{k^2}\sum_{l\geq 1}\frac{\omega_l}{b^l} + \frac{k^2}{t}\sum_{l\geq 1}\frac{\omega_l'}{b^l} + \sum_{l\geq 1}\frac{\omega_l''}{b^l} \qquad (17)\end{aligned}$$

where $\omega_l, \omega_l', \omega_l''$ are differential forms of degree (l,l), depending on M but not on k. Thus $\bar\beta_{t,M}$ has the form

$$\bar\beta_{t,M} = \sum_{k\in\mathbb{Z},k|c} \left(\frac{-b}{\pi\sqrt{t}}\right)^d \frac{1}{\sqrt{\det C}(1+\delta_{r_0=u_0})} \left(\sum_{l\in\mathbb{Z}} (\frac{t}{k^2})^l \alpha_l(b)\right)$$
$$\cdot\left(\exp\left(-\frac{\pi^2c^2}{k^2}t\,{}^tu_0Yu_0 - \frac{k^2}{t}{}^tr_0Yr_0\right) + \exp\left(-\frac{\pi^2c^2}{k^2}t\,{}^tr_0Yr_0 - \frac{k^2}{t}{}^tu_0Yu_0\right)\right)$$

with $\alpha_l(b)$ being a differential form of degrees greater or equal to $|l|$, with coefficients in polynomials in $1/b$. In particular, the sum over l is finite. Now for $a,b\in\mathbb{R}^+$, $\alpha\in\mathbb{R}$ the Bessel K-functions provide the formula

$$\frac{1}{\Gamma(s)}\int_0^\infty e^{-at-b/t}t^{s-1-\alpha}dt = \frac{2}{\Gamma(s)}\sqrt{\frac{a}{b}}^{\alpha-s} K(\alpha-s, 2\sqrt{ab})$$

and thus

$$\frac{\partial}{\partial s}_{|s=0}\left(\frac{1}{\Gamma(s)}\int_0^\infty e^{-at-b/t}t^{s-1-\alpha}dt\right) = 2\sqrt{\frac{a}{b}}^{\alpha} K(\alpha, 2\sqrt{ab})\ .$$

We define

$$\|M,Y\| := \sqrt{{}^trYr\cdot{}^tuYu}\ ;$$

by Proposition 6.1 we know that this value does not depend on the choice of r and u. More easily, one can verify this using $\|M,Y\|^2 + \langle M,Y\rangle^2 = 2\mathrm{Tr}\,MYMY$. Also we set

$$\rho(r_0,u_0) := \sqrt{\frac{{}^tr_0Yr_0}{{}^tu_0Yu_0}}\ .$$

Hence we find for the derivative at $s=0$ of the Mellin transform of $\bar\beta_{t,M}$

$$\frac{\partial}{\partial s}_{|s=0}\left(\frac{1}{\Gamma(s)}\int_0^\infty \bar\beta_{t,M}t^{s-1}dt\right)$$
$$= \sum_{k\in\mathbb{Z},k|c}\sum_{l\in\mathbb{Z}} \alpha_l(b)|k|^{-2l}\left(\frac{-b}{\pi}\right)^d \frac{1}{\sqrt{\det C}(1+\delta_{r_0=u_0})}$$
$$\cdot 2\left(\sqrt{\frac{\pi^2c^2\cdot{}^tu_0Yu_0}{k^4\cdot{}^tr_0Yr_0}}^{d/2-l} + \sqrt{\frac{\pi^2c^2\cdot{}^tr_0Yr_0}{k^4\cdot{}^tu_0Yu_0}}^{d/2-l}\right)$$
$$\cdot K(d/2-l, 2\sqrt{\pi^2\|M,Y\|^2})$$
$$= \sum_{k\in\mathbb{Z},k|c}\sum_{l\in\mathbb{Z}} \alpha_l(b)\frac{2c^{d/2-l}(-b)^d}{\pi^{l+d/2}|k|^d\sqrt{\det C}(1+\delta_{r_0=u_0})}$$
$$\cdot\left(\rho(r_0,u_0)^{l-d/2} + \rho(r_0,u_0)^{d/2-l}\right) K(d/2-l, 2\pi\|M,Y\|)$$

$$= \sum_{l\in\mathbb{Z}} \alpha_l(b) \frac{2(\pi c)^{-d/2-l}(-b)^d \sigma_d(c)}{\sqrt{\det C}(1+\delta_{r_0=u_0})}$$
$$\cdot \Big(\rho(r_0,u_0)^{l-d/2} + \rho(r_0,u_0)^{d/2-l}\Big) K(d/2-l, 2\pi\|M,Y\|)$$

with $\sigma_m(c) := \sum_{k\in\mathbb{Z}^+, k|c} k^m$ being the divisor function. For $c = \prod_{p \text{ prime}} p^{\nu_p}$, one finds

$$\sigma_m(c) = c^m \prod_{p \text{ prime}} \frac{1-p^{-m(\nu_p+1)}}{1-p^{-m}}$$

and thus $\sigma_m(c) \in]c^m, \zeta(m)c^m[$. The form γ is given by the linear term in b in the above equation, for which $|l| \le d-1$. Set

$$\eta(r_0,u_0) := e^{-2\pi\|M,Y\|}\rho(r_0,u_0)^{-d/2} \exp\Big(- c\rho(r_0,u_0)\cdot {}^t u_0(C^{-1}-\pi Y)u_0$$
$$-2\pi c\, {}^t r_0 Y D C^{-1} u_0 - \pi^2 c\rho(r_0,u_0)^{-1}\cdot {}^t r_0(Y(C-DC^{-1}D)Y - \frac{1}{\pi}Y)r_0\Big)$$
$$= \rho(r_0,u_0)^{-d/2} \exp\Big(- c\rho(r_0,u_0)\cdot {}^t u_0 C^{-1} u_0 - 2\pi c\, {}^t r_0 Y D C^{-1} u_0$$
$$-\pi^2 c\rho(r_0,u_0)^{-1}\cdot {}^t r_0 Y(C-DC^{-1}D)Y r_0\Big).$$

The Bessel K-functions have for $|x|\to\infty$ the asymptotics

$$K(v,x) = \sqrt{\frac{\pi}{2x}} e^{-x}\big(1+O(\frac{1}{x})\big)$$

and thus we find for $\|M,Y\|\to\infty$ by setting $t := \frac{k^2}{c\pi}\rho(r_0,u_0)$ in the defining equation for the α_l

$$\frac{\partial}{\partial s}_{|s=0}\left(\frac{1}{\Gamma(s)}\int_0^\infty \bar\beta_{t,M} t^{s-1} dt\right)$$
$$= \frac{(\pi c)^{-d/2}(-b)^d\sigma_d(c)}{\sqrt{\|M,Y\|\det C}(1+\delta_{r_0=u_0})}\big(\eta(r_0,u_0)+\eta(u_0,r_0)\big)\big(1+O(\frac{1}{\|M,Y\|})\big).$$

For d odd the Bessel K-functions have special values, and one thus finds explicit expressions for the Fourier coefficients similar to (18). Using the *polylogarithm* defined for $|q|<1$, $l\in\mathbb{R}$ by

$$\mathrm{Li}_l(q) := \sum_{k=1}^\infty \frac{q^k}{k^l} \tag{18}$$

we have the equality for $m\in\mathbb{Z}^+$

$$\sum_{c=1}^\infty \frac{\sigma_m(c)}{c^l} q^c = \sum_{n=1}^\infty n^{m-l}\mathrm{Li}_l(q^n).$$

Thus we obtain with

$$q(r_0,u_0) := \exp\Big(-\rho(r_0,u_0)\cdot {}^t u_0 C^{-1} u_0 - 2\pi {}^t r_0 Y D C^{-1} u_0 \\ -\pi^2 \rho(r_0,u_0)^{-1}\cdot {}^t r_0 Y (C - DC^{-1}D) Y r_0 - 2\pi i {}^t u_0 X r_0 \Big)$$

the following

Lemma 6.2. *When summing the part of the Fourier expansion corresponding to frequency matrices which have the same pair of primitive vectors* r_0, u_0*, we obtain with* $M_0 := \frac{1}{2}(r_0 \cdot^t u_0 + u_0 \cdot^t r_0)$

$$\sum_{c\in\mathbb{Z}^+} e^{-2\pi i \langle cM_0, X\rangle} \frac{\partial}{\partial s}_{|s=0} \left(\frac{1}{\Gamma(s)} \int_0^\infty \bar{\beta}_{t,cM_0} t^{s-1} dt \right)$$

$$= \frac{\pi^{-d/2}(-b)^d \rho(r_0,u_0)^{-d/2}}{\sqrt{\|M_0,Y\| \det C}(1+\delta_{r_0=u_0})} \cdot \sum_{n=1}^\infty n^{\frac{d-1}{2}} \Big(\mathrm{Li}_{\frac{d+1}{2}}\big(q(r_0,u_0)^n\big) + O\big(\frac{1}{\|M_0,Y\|}\big) \mathrm{Li}_{\frac{d+3}{2}}\big(q(r_0,u_0)^n\big)\Big)$$

$+$ *this same term with* r_0, u_0 *exchanged*

$$= \frac{\pi^{-d/2}(-b)^d}{\sqrt{\|M_0,Y\| \det C}(1+\delta_{r_0=u_0})} \sum_{n=1}^\infty n^{\frac{d-1}{2}} \Bigg(\rho(r_0,u_0)^{-d/2} \mathrm{Li}_{\frac{d+1}{2}}\big(q(r_0,u_0)^n\big) \\ + \rho(r_0,u_0)^{d/2} \mathrm{Li}_{\frac{d+1}{2}}\big(q(u_0,r_0)^n\big) \Bigg) \cdot \left(1 + O\big(\frac{1}{\|M_0,Y\|}\big) \right).$$

Here polylogarithms of forms have to be interpreted via the power series in equation (18).

6.2 The coefficient of the constant term

For $M = 0$ we find by applying again the Poisson summation formula to both sums

$$\bar{\beta}_{t,0} = \sum_{r\in\mathbb{Z}^d} \left(\frac{-b}{\pi\sqrt{t}}\right)^d \frac{1}{\sqrt{\det C}} \exp\Big(-\frac{\pi}{t} {}^t r Y (C - DC^{-1}D) Y r \Big) \tag{19}$$

$$+ \sum_{u\in\mathbb{Z}^d} \left(\frac{-b}{\pi\sqrt{t}}\right)^d \frac{1}{\sqrt{\det C}} \exp\Big(-\pi t {}^t u C^{-1} u \Big) - \left(\frac{-b}{\pi\sqrt{t}}\right)^d \frac{1}{\sqrt{\det C}}$$

$$= \sum_{\hat{r}\in\mathbb{Z}^d} \left(\frac{-b}{\pi}\right)^d \frac{\exp\Big(-\pi t {}^t \hat{r} Y^{-1} (C - DC^{-1}D)^{-1} Y^{-1} \hat{r} \Big)}{\sqrt{\det(C(C - DC^{-1}D))} \det Y}$$

$$+ \sum_{\hat{u}\in\mathbb{Z}^d} \left(\frac{-b}{\pi t}\right)^d \exp\Big(-\frac{\pi}{t} {}^t \hat{u} C \hat{u} \Big) - \left(\frac{-b}{\pi\sqrt{t}}\right)^d \frac{1}{\sqrt{\det C}}. \tag{20}$$

Using (16), we find

$$\det(Y^2C(C - DC^{-1}D)) = \det(YC + YD)^2$$

and (by Corollary 3.5)

$$\frac{1}{\det(\pi Y(C+D))} = \det\left(1 + \frac{1}{2\pi i b}\Omega^E\right) = \sum_{j=0}^{d}(-b)^{-j}c_j(\overline{E}),$$

and thus (20) simplifies to

$$\bar{\beta}_{t,0} = \theta_1(t) + \theta_2(t) - \left(\frac{-b}{\pi\sqrt{t}}\right)^d \frac{1}{\sqrt{\det C}} \tag{21}$$

where

$$\theta_1(t) := \sum_{\hat{r}\in\mathbb{Z}^d}(-b)^d \det\left(1 + \frac{1}{2\pi i b}\Omega^E\right)\exp\Big(-\pi t\cdot {}^t\hat{r}Y^{-1}(C\pm D)^{-1}Y^{-1}\hat{r}\Big),$$

$$\theta_2(t) := \sum_{\hat{u}\in\mathbb{Z}^d}\left(\frac{-b}{\pi t}\right)^d \exp\Big(-\frac{\pi}{t}\cdot {}^t\hat{u}C\hat{u}\Big).$$

Note that the term $-\left(\frac{-b}{\pi\sqrt{t}}\right)^d\frac{1}{\sqrt{\det C}}$ vanishes under Mellin transformation [K1, Remark on p.12]. The b-linear term of the second summand $\theta_2(t)$ in (21) is

$$\theta_2(t)^{[b]} = \frac{1}{(d-1)!}\sum_{u\in\mathbb{Z}^d}\left(\frac{-1}{\pi t}\right)^d \exp\Big(-\frac{1}{t}{}^tuY^{-1}u\Big)\left(\frac{1}{2\pi i t}{}^tuY^{-1}\Omega^E u\right)^{\wedge(d-1)}$$

with Mellin transform

$$Z_2(s)^{[b]} := -\frac{\Gamma(2d-1-s)}{\Gamma(s)(d-1)!}\sum_{u\in\mathbb{Z}^d}\left(\frac{-1}{\pi}\right)^d({}^tuY^{-1}u)^{1-2d+s}\left(\frac{1}{2\pi i}{}^tuY^{-1}\Omega^E u\right)^{\wedge(d-1)},$$

and thus the corresponding summand of γ equals

$$Z_2'(0) = \frac{(2d-2)!}{(d-1)!\pi^d}\sum_{u\in\mathbb{Z}^d\setminus\{0\}}({}^tuY^{-1}u)^{1-2d}\left(\frac{-1}{2\pi i}{}^tuY^{-1}\Omega^E u\right)^{\wedge(d-1)}.$$

This term is homogeneous in Y of degree $2-d$; thus it behaves like $|Y|^{2-d}$ for $|Y|\to\infty$ or $|Y|\to 0$.

By proceeding as in (17) we observe that the first summand $\theta_1(t)$ in (21) has the form

$$
\begin{aligned}
&\theta_1(t) = \\
&\quad \sum_{r\in\mathbb{Z}^d} (-b)^d \det\left(1+\frac{1}{2\pi ib}\Omega^E\right)\exp\Big(-\pi t{}^t rY^{-1}(C-DC^{-1}D)^{-1}Y^{-1}r\Big) \\
&= \sum_{r\in\mathbb{Z}^d} (-b)^d \det\left(1+\frac{1}{2\pi ib}\Omega^E\right)\exp\Big(-\pi^2 t{}^t rY^{-1}r\Big) \\
&\qquad\qquad \cdot\exp\Big(-\pi t{}^t rY^{-1}\Big((C-DC^{-1}D)^{-1}-\pi Y\Big)Y^{-1}r\Big) \\
&= \sum_{r\in\mathbb{Z}^d} (-b)^d \det\left(1+\frac{1}{2\pi ib}\Omega^E\right)\exp\Big(-\pi^2 t{}^t rY^{-1}r\Big) \\
&\qquad\qquad \cdot\left(1+\sum_{k=1}^{d}\sum_{\ell=1}^{k} t^\ell(-b)^{-k}\omega_{k,\ell}\right)
\end{aligned}
$$

with $\omega_{k,\ell}$ being a (k,k)-form, homogeneous in Y of degree $-\ell-2k$ and homogeneous in r of degree 2ℓ. The coefficient of b in θ_1 is given by

$$
\theta_1(t)^{[b]} = \theta_{11}(t) + \theta_{12}(t)
$$

where

$$
\begin{aligned}
\theta_{11}(t) &:= -\sum_{r\in\mathbb{Z}^d}\exp\Big(-\pi^2 t\cdot{}^t rY^{-1}r\Big)c_{d-1}(\overline{E})\,, \\
\theta_{12}(t) &:= -\sum_{r\in\mathbb{Z}^d}\exp\Big(-\pi^2 t\cdot{}^t rY^{-1}r\Big)\sum_{k=1}^{d-1}\sum_{\ell=1}^{k} t^\ell\omega_{k,\ell}c_{d-1-k}(\overline{E})\,.
\end{aligned}
$$

The Mellin transform of this term thus equals

$$
\begin{aligned}
Z_{11}(s)c_{d-1}(\overline{E}) + \frac{1}{\Gamma(s)}Z_{12}(s) &:= \sum_{r\in\mathbb{Z}^d\setminus\{0\}}\Big(\pi^2\cdot{}^t rY^{-1}r\Big)^{-s}c_{d-1}(\overline{E}) \\
+ \sum_{r\in\mathbb{Z}^d\setminus\{0\}}\sum_{k=1}^{d-1}\sum_{\ell=1}^{k}&\frac{\Gamma(s+\ell)}{\Gamma(s)}\Big(\pi^2\cdot{}^t rY^{-1}r\Big)^{-s-\ell}\omega_{k,\ell}c_{d-1-k}(\overline{E})
\end{aligned}
$$

which is homogeneous in Y of degree $2-2d+s$. In particular the Mellin transform of θ_1 converges in (21) for $\mathrm{Re}\, s > d/2$ when subtracting the $\hat{r}=0$ summand (and similarly in (19) for Re $s<0$ when subtracting the $r=0$ summand). Notice that $\theta_{22}(t)\to 0$ for $t\to\infty$ and thus $\frac{1}{\Gamma(s)}Z_{22}(s)\to 0$ for $s\to 0$. Hence $\frac{\partial}{\partial s}_{|s=0}\frac{1}{\Gamma(s)}Z_{12}(s) = Z_{12}(0)$. Furthermore $Z_{11}(0)=-1$. Clearly for $\alpha\in\mathbb{R}^+$

$$
Z_{11}(s)_{|\alpha Y} = \alpha^s Z_{11}(s)_{|Y}
$$

and thus

$$
Z'_{11}(0)_{|\alpha Y} = -\log\alpha + Z'_{11}(0)_{|Y}\,.
$$

Concluding we find

Theorem 6.3. *The differential form γ representing the torsion form verifies for $|Y| \to \infty$*

$$\gamma = Z'_{11}(0) c_{d-1}(\overline{E}) + Z_{12}(0) + Z'_2(0) + O(e^{-c|Y|}) \qquad (22)$$

where Z_{11} is a classical real-valued Epstein zeta function; Z_{12} is a sum of Epstein zeta functions with polynomials in the numerator; and $Z'_2(0)$ is given by a convergent series. The first term in (22) behaves like $-\log|Y| \cdot |Y|^{2-2d} c_1 + |Y|^{2-2d} c_2$, the second term is homogeneous in Y of degree $2-2d$ and the third term is homogeneous in Y of degree $2-d$.

References

[B3] Bismut, J.-M.: *Equivariant immersions and Quillen metrics.* J. Diff. Geom. **41**, 53-157 (1995).

[B4] Bismut, J.-M.: Holomorphic families of immersions and higher analytic torsion forms. Astérisque **244** :SMF Paris 1997.

[BM] Bismut, J.-M., Ma, Xiaonan: *Holomorphic immersions and equivariant torsion forms.* To appear in J. reine angew. Math.

[BBM] Berthelot, P., Breen, L., Messing, W., Théorie de Dieudonné Cristalline II (Lecture Notes Math. **930**): Springer 1982.

[Bo] Bost, J.-B.:*Intersection theory on arithmetic surfaces and L^2_1 metrics*, Letter dated March 6th, 1998.

[BKK] Burgos, Jose I., Kramer, J., Kühn, U.: *Cohomological Arithmetic Chow groups.* e-print http://www.arxiv.org/abs/math.AG/ 0404122 .

[E] P. Epstein, *Zur Theorie allgemeiner Zetafunktionen*, Math. Ann. **56**, 615–644 (1903).

[Fal] Faltings, G.: Lectures on the arithmetic Riemann-Roch theorem: Princeton University Press 1992.

[FC] Faltings, G., Chai, Ching-Li: Degeneration of abelian varieties. With an appendix by David Mumford (Erg. Math. **22**): Springer 1990.

[GS2] Gillet, H., Soulé, C.: *Arithmetic intersection theory.* Publ. Math. I.H.E.S. **72**, 94–174 (1990).

[GS3] Gillet, H., Soulé, C.: *Characteristic classes for algebraic vector bundles with hermitian metrics I, II.* Ann. of Math. **131**, 163-203, 205-238 (1990).

[GS8] Gillet, H., Soulé, C.: *An arithmetic Riemann-Roch theorem.* Invent. Math. **110**, 473-543 (1992).

[G] van der Geer, G.: *Cycles on the moduli space of abelian varieties.* 65–89 in: Faber, C., Looijenga, E.: Moduli of curves and abelian varieties. Aspects Math. **E33**, Vieweg 1999.

[Hi1] Hirzebruch, F.: Collected Papers vol. I: Springer 1987.

[Hi2] Hirzebruch, F.: Collected Papers vol. II: Springer 1987.

[Hi3] Hirzebruch, F.: Topological Methods in Algebraic Geometry: Springer 1978.

[KK] Kaiser, Ch., Köhler, K.: *A fixed point formula of Lefschetz type in Arakelov geometry III: representations of Chevalley schemes and heights of flag varieties.* to appear in Inv. Math.

[KS] Keel, S., Sadun, L.: *Oort's conjecture for $A_g \otimes \mathbb{C}$.* J. Am. Math. Soc. **16**, 887–900 (2003).

[K1] Köhler, K.: *Torus fibrations and moduli spaces*, 166–195 in: Reznikov, A., Schappacher, N., eds.: Regulators in Analysis, Geometry and Number Theory. Progress in Math. **171**, Birkhäuser 2000.

[K2] Köhler, K.: *A Hirzebruch proportionality principle in Arakelov geometry*, prépublication Jussieu **284**, avril 2001.

[KR1] Köhler, K., Roessler, D.: *A fixed point formula of Lefschetz type in Arakelov geometry I: statement and proof.* Inv. Math. **145**, 333-396 (2001).

[KR2] Köhler, K., Roessler, D.: *A fixed point formula of Lefschetz type in Arakelov geometry II: a residue formula.* Ann. Inst. Fourier **52**, 1–23 (2002).

[KR4] Köhler, K., Roessler, D.: *A fixed point formula of Lefschetz type in Arakelov geometry IV: abelian varieties.* J. reine angew. Math. **556**, 127–148 (2003).

[Kü] Kühn, U.,: *Generalized arithmetic intersection numbers.* J. reine angew. Math. **534**, 209–236 (2001).

[MR] Maillot, V., Roessler, D.: *Conjectures sur les dérivées logarithmiques des fonctions L d'Artin aux entiers négatifs.* Math. Res. Lett. **9**, 715–724 (2002).

[MiS] Milnor, J. W., Stasheff, J. D.: Characteristic Classes: Ann. Math. Stud. **76**, Princeton Univ. Press 1974.

[M] Mumford, D.: *Hirzebruch Proportionality Theorem in the Non Compact Case.* Inv. Math. **42**, 239–272 (1977).

[R1] Roessler, D.:*Lambda-structure on Grothendieck groups of Hermitian vector bundles.* Israel J. Math. **122**, 279–304 (2001).

[R2] Roessler, D.: *An Adams-Riemann-Roch theorem in Arakelov geometry.* Duke Math. J. **96** 61–126 (1999)

[S] Soulé, C.: *Hermitian vector bundles on arithmetic varieties.* In: Algebraic geometry, Santa Cruz 1995, 383–419, Proc. Sympos. Pure Math. **62**, Part 1, AMS 1997.

[SABK] Soulé, C., Abramovich, D., Burnol, J. F. and Kramer, J.: Lectures on Arakelov Geometry (Cambridge studies in math. 33): Cambridge university press 1992.

[T] Tamvakis, H.: *Arakelov theory of the Lagrangian Grassmannian.* J. reine angew. Math. **516**, 207-223 (1999).

[Y] Yoshikawa, K.: *Discriminant of theta divisors and Quillen metrics.* J. Differential Geom. **52**, 73–115 (1999).